Dynamic Programming

- Steps.
  - ✓ View the problem solution as the result of a sequence of decisions.
  - ✓ Obtain a formulation for the problem state.
  - ✓ Verify that the principle of optimality holds.
  - ✓ Set up the dynamic programming recurrence equations.
  - ✓ Solve these equations for the value of the optimal solution.
  - Perform a traceback to determine the optimal solution.

Dynamic Programming

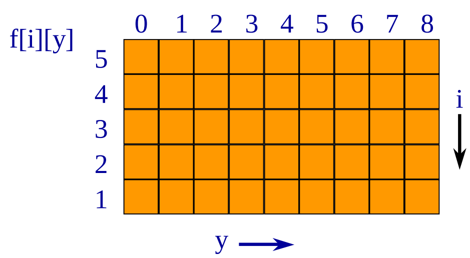
- When solving the dynamic programming recurrence recursively, be sure to avoid the recomputation of the optimal value for the same problem state.
- To minimize run time overheads, and hence to reduce actual run time, dynamic programming recurrences are almost always solved iteratively (no recursion).

0/1 Knapsack Recurrence

- If $w_n \le y$, $f(n,y) = p_n$.
- If $w_n > y$, $f(n,y) = 0$.
- When $i < n$
  - $f(i,y) = f(i+1,y)$ whenever $y < w_i$.
  - $f(i,y) = \max\{f(i+1,y), f(i+1,y-w_i) + p_i\}$, $y >= w_i$.
- Assume the weights and capacity are integers.
- Only $f(i,y)$s with $1 <= i <= n$ and $0 <= y <= c$ are of interest.

Iterative Solution Example

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,7,10,9,3]



Compute f[5][*]

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,7,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4									
3									
2									
1									

i ↓, y →

Compute f[4][*]

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3									
2									
1									

i ↓, y →

$f(i,y) = \max\{f(i+1,y), f(i+1,y-w_i) + p_i\}, y >= w_i$

Compute f[3][*]

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2									
1									

i ↓, y →

$f(i,y) = \max\{f(i+1,y), f(i+1,y-w_i) + p_i\}, y >= w_i$

Compute f[2][*]

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									

i ↓, y →

$f(i,y) = \max\{f(i+1,y), f(i+1,y-w_i) + p_i\}, y >= w_i$

Compute f[1][c]

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]

i \ y	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

i ↓   y →

$f(i,y) = \max\{f(i+1,y), f(i+1,y-w_i) + p_i\}, y >= w_i$

Iterative Implementation

```
// initialize f[n][]
int yMax = Math.min(w[n] - 1, c);
for (int y = 0; y <= yMax; y++)
   f[n][y] = 0;
for (int y = w[n]; y <= c; y++)
   f[n][y] = p[n];
```

Iterative Implementation

```
// compute f[i][y], 1 < i < n
for (int i = n - 1; i > 1; i--)
{
  yMax = Math.min(w[i] - 1, c);
  for (int y = 0; y <= yMax; y++)
    f[i][y] = f[i + 1][y];
  for (int y = w[i]; y <= c; y++)
    f[i][y] = Math.max(f[i + 1][y],
               f[i + 1][y - w[i]] + p[i]);
}
```

Iterative Implementation

```
 // compute f[1][c]
 f[1][c] = f[2][c];
 if (c >= w[1])
   f[1][c] = Math.max(f[1][c],
          f[2][c-w[1]] + p[1]);
}
```

Time Complexity

- O(cn).
- Same as for the recursive version with no recomputations.
- Iterative version is expected to run faster because of lower overheads.
  - No checks to see if f[i][j] already computed (but all f[i][j] are computed).
  - Method calls replaced by for loops.

Traceback

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

y →, i ↓

f[1][8] = f[2][8] => x_1 = 0

Traceback

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

y →, i ↓

f[2][8] != f[3][8] => x_2 = 1

Traceback

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

y →, i ↓

f[3][5] != f[4][5] => x_3 = 1

Traceback

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]

i \ y	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

i ↓ y →

f[4][0] = f[5][0] => x_4 = 0

Traceback

- n = 5, c = 8, w = [4,3,5,6,2], p = [9,8,10,9,3]

f[i][y]

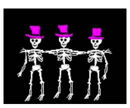
i \ y	0	1	2	3	4	5	6	7	8
5	0	0	3	3	3	3	3	3	3
4	0	0	3	3	3	3	9	9	12
3	0	0	3	3	3	10	10	13	13
2	0	0	3	8	8	11	11	13	18
1									18

i ↓ y →

f[5][0] = 0 => x_5 = 0

Complexity Of Traceback

- O(n)



Matrix Multiplication Chains

- Multiply an m x n matrix A and an n x p matrix B to get an m x p matrix C.

$$C(i,j) = \sum_{k=1}^{n} A(i,k) * B(k,j)$$

- We shall use the number of multiplications as our complexity measure.
- n multiplications are needed to compute one C(i,j).
- mnp multiplicatons are needed to compute all mp terms of C.

Matrix Multiplication Chains

- Suppose that we are to compute the product X*Y*Z of three matrices X, Y and Z.
- The matrix dimensions are:
  - X:(100 x 1), Y:(1 x 100), Z:(100 x 1)
- Multiply X and Y to get a 100 x 100 matrix T.
  - 100 * 1 * 100 = 10,000 multiplications.
- Multiply T and Z to get the 100 x 1 answer.
  - 100 * 100 * 1 = 10,000 multiplications.
- Total cost is 20,000 multiplications.
- 10,000 units of space are needed for T.

Matrix Multiplication Chains

- The matrix dimensions are:
  - X:(100 x 1)
  - Y:(1 x 100)
  - Z:(100 x 1)
- Multiply Y and Z to get a 1 x 1 matrix T.
  - 1 * 100 * 1 = 100 multiplications.
- Multiply X and T to get the 100 x 1 answer.
  - 100 * 1 * 1 = 100 multiplications.
- Total cost is 200 multiplications.
- 1 unit of space is needed for T.

Product Of 5 Matrices

- Some of the ways in which the product of 5 matrices may be computed.
  - A*(B*(C*(D*E))) right to left
  - (((A*B)*C)*D)*E left to right
  - (A*B)*((C*D)*E)
  - (A*B)*(C*(D*E))
  - (A*(B*C))*(D*E)
  - ((A*B)*C)*(D*E)

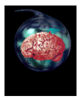
Find Best Multiplication Order

- Number of ways to compute the product of q matrices is $O(4^q/q^{1.5})$.

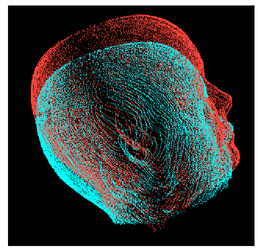
- Evaluating all ways to compute the product takes $O(4^q/q^{0.5})$ time.

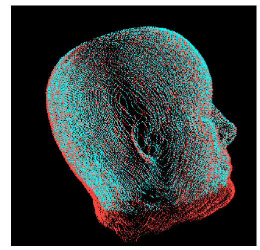

An Application

- Registration of pre- and post-operative 3D brain MRI images to determine volume of removed tumor.



3D Registration





3D Registration

- Each image has 256 x 256 x 256 voxels.
- In each iteration of the registration algorithm, the product of three matrices is computed at each voxel … (12 x 3) * (3 x 3) * (3 x 1)
- Left to right computation => 12 * 3 * 3 + 12 * 3*1 = 144 multiplications per voxel per iteration.
- 100 iterations to converge.

3D Registration

- Total number of multiplications is about $2.4 * 10^{11}$.
- Right to left computation => 3 * 3*1 + 12 * 3 * 1 = 45 multiplications per voxel per iteration.
- Total number of multiplications is about $7.5 * 10^{10}$.
- With 10^8 multiplications per second, time is 40 min vs 12.5 min.