

Connected Dominating Sets in Disk Graphs with Bidirectional Links

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Abstract

In this paper, we study the Connected Dominating Set (CDS) problem in disk graphs. The CDS problem has a significant impact on an efficient design of routing protocols in wireless networks. This problem has been studied extensively in unit disk graphs, in which each node has the same transmission range. However, in wireless ad hoc networks, the transmission ranges of all nodes are not necessary equal. In this paper, we introduce the CDS problem in disk graphs and present a constant approximation algorithm which can be implemented as a distributed algorithm.

Index Terms

Connected Dominating Set, Independent Set, Disk Graphs, Wireless Ad Hoc Networks

I. INTRODUCTION

IN wireless ad hoc networks, there is no fixed or pre-defined infrastructure. Nodes in wireless networks communicate via a shared medium, either through a single hop or multihops. Although there is no physical backbone infrastructure, a virtual backbone can be formed by constructing a Connected Dominating Set (CDS). Given an undirected graph $G = (V, E)$, a subset $V' \subseteq V$ is a CDS of G if for each node $v \in V$, v is either in V' or adjacent to some node in V' and the subgraph induced by V' is connected. With the help of the CDS, routing is easier and can adapt quickly to topology changes of a network. To reduce the traffic during

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communication and simplify the connectivity management, it is desirable to construct a Minimum CDS (MCDS).

The MCDS problem has been studied intensively in Unit Disk Graphs (UDG), in which each node has the same transmission range. The MCDS problem in UDG has been shown to be NP-hard [1]. To build a CDS, most of current algorithms first find a Maximal Independent Set (MIS) I of G and then connect all nodes in I to have a CDS. The most relevant related work are in [2], [3]. In [2], Wan *et al.* proposed the first distributed algorithm with a performance ratio of 8. Later, Li *et al.* proposed a better algorithm with a performance ratio of $(4.8 + \ln 5)$ by constructing a Steiner tree when connecting all nodes in I [3].

However, in practice, the transmission ranges of all nodes are not necessary equal. In this case, an ad hoc network can be modeled using a directed graph $G = (V, E)$. The nodes in V are located in a Euclidean plane and each node $v_i \in V$ has a transmission range $r_i \in [r_{min}, r_{max}]$. A directed edge $(v_i, v_j) \in E$ if and only if $d(i, j) \leq r_i$ where $d(i, j)$ denotes the Euclidean distance between v_i and v_j . Such graphs are called disk graphs. An edge (v_i, v_j) is bidirectional if both (v_i, v_j) and (v_j, v_i) are in E , i.e., $d(i, j) \leq \min\{r_i, r_j\}$. In other words, two nodes v_i and v_j are adjacent if v_i is in the disk D_j centered at v_j with radius r_j and v_j is in the disk D_i centered at v_i with radius r_i . In this paper, we only study the MCDS problem in disk graphs where all the edges in the network are bidirectional. In this case, G is undirected.

The MCDS problem in Disk Graphs with Bidirectional links (DGB) is NP-hard since the MCDS problem in UDG is NP-hard. In this paper, we present a constant approximation algorithm for computing a minimum CDS in DGB. This algorithm is a centralized version but it can be implemented as a distributed algorithm. We also show a size relationship between an MIS and a CDS in DGB. The analysis of this relationship can help us to study the CDS problem in general disk graphs, where both unidirectional and bidirectional links are considered.

II. AN APPROXIMATION ALGORITHM AND ANALYSIS

In this section, we present a constant approximation algorithm for the CDS problem in DGB and analyze its performance ratio. Let $k = \frac{r_{max}}{r_{min}}$. We first introduce the following lemma:

Lemma 1: In a DGB, every node is adjacent to at most K independent nodes, where $K = 5$ if $k = 1$, otherwise, $K = 6(3\lceil \log_2 k \rceil + 2)$.

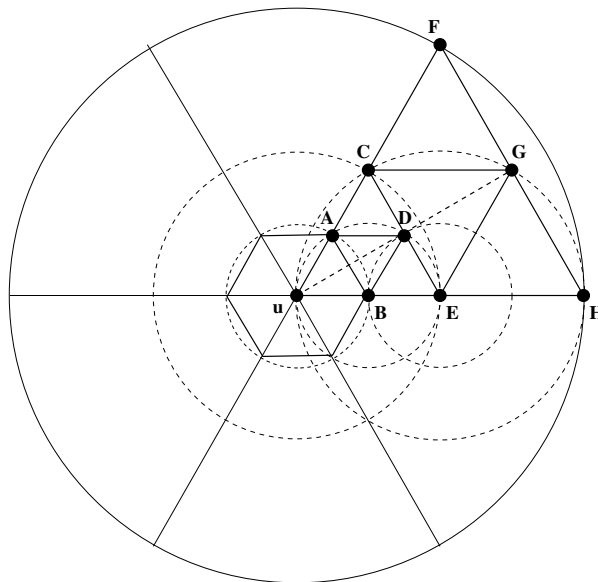


Fig. 1. On the Proof of Maximum Number of Independent Neighbors

Proof: When $k = 1$, a DGB is a UDG. Thus the lemma holds. When $k > 1$, choose a node u with radius $r_u = r_{max}$, i.e. node u has the largest transmission range. Taking u as the center, draw a number of regular triangles as shown in Figure 1. First, draw a disk centered at u with radius $uB = r_{min}$. Place 6 regular triangles with edge length r_{min} around u . Second, draw a disk centered at u with radius $uE = 2r_{min}$. Place 18 regular triangles with edge length r_{min} . Figure 1 shows 3 of such 18 triangles, $\triangle ABD$, $\triangle ACD$, and $\triangle BDE$. Next, draw a disk centered at u with radius $uH = 4r_{min}$. Place 18 regular triangles with edge length $2r_{min}$. Figure 1 shows 3 of these 18 triangles, $\triangle CEG$, $\triangle CFG$, and $\triangle EGH$. In this case, since $r_{max} = 4r_{min}$, we stop. If $r_{max} > 4r_{min}$, we can draw another disk centered at u with radius $8r_{min}$ and place 18 regular

triangles with edge length $4r_{min}$. In general, for $l \in [1, \lceil \log_2 k \rceil]$, draw 18 regular triangles with edge length $2^{l-1}r_{min}$. Thus the total number of triangles is $6 + 18\lceil \log_2 k \rceil$. Now we show that every pair of nodes v_i and v_j that are adjacent to u and lie in *the same triangle* is connected. It is trivial for the regular triangles with edge length r_{min} since $d(v_i, v_j) \geq r_{min}$. Assume that v_i, v_j lie in the triangle $\triangle EGH$, we have r_i and r_j greater than or equal to $2r_{min}$. Otherwise, r_i and r_j cannot be adjacent to u . But $d(v_i, v_j) \leq 2r_{min}$, hence v_i and v_j are connected. In general, let v_i and v_j lie in the same triangle with edge length $2^{l-1}r_{min}$, r_i and r_j must be greater than or equal to $2^{l-1}r_{min}$ in order to connect to u . But $d(v_i, v_j) \leq 2^{l-1}r_{min}$, hence v_i and v_j are connected.

Now, we prove that there exist two nodes v_i and v_j lying in two *different triangles* such that v_i and v_j are independent. Without loss of generality, let v_i lie in $\triangle CFG$ and v_j lie in $\triangle EGH$. Let node v_i lie at vertex C . Then a part of $\triangle EGH$ is not inside the disk centered at v_i with radius uC . Let choose v_j in that part. It follows that v_i and v_j are independent.

Note that after placing all triangles around u , we have 6 arcs left nearby the boundary of the disk centered at u with radius r_{max} . Figure 1 shows one of 6 arcs, that is, an arc FH . We claim that all nodes lying in this arc form a clique. Let v_i and v_j lie in this arc. Then r_i and r_j of corresponding disks must be greater than or equal to $4r_{min}$. Note that $FH = 4r_{min}$. Hence $d(v_i, v_j) \leq 4r_{min}$. This follows that all nodes in each arc form a clique.

We have $6 + 18\lceil \log_2 k \rceil + 6$ areas. Hence, the maximum number of independent neighbors of a node in DGB is at most $6(3\lceil \log_2 k \rceil + 2)$ when $k > 1$.

■

A. Algorithms

Our algorithm has two phases. First, we construct a maximal independent set I that satisfies this property:

Lemma 2: Any pair of complementary subsets of the MIS has a distance of exactly two hops.

The algorithm in [2] satisfies Lemma 2. Hence we can use this algorithm for our first phase.

In the second phase, we need to find a set of connectors B that can connect all nodes in I . Let a black node be a node in I and a blue node represent a node in B . Since in the DGB, any node is adjacent to at most K independent nodes, we can find a set of blue nodes given a set I of black nodes by using a Steiner tree, which is a tree interconnecting all nodes in I . The nodes in the Steiner tree but not in I are called Steiner nodes. To reduce the size of a CDS, we need to find a Steiner tree with a Minimum number of Steiner Nodes (MSN). We can define this problem as follows:

Definition 1: Steiner Tree with MSN: Given a graph $G = (V, E)$ and a set of nodes $V' \subseteq V$ called *terminals*, construct a Steiner tree T for V' such that the number of Steiner nodes is minimum.

Define a *black-blue component* as a connected component of the subgraph induced only by black and blue nodes, ignoring connections between blue nodes. Given a maximal independent set I of G , we can interconnect all nodes in I as follows. We first color all nodes in G but not in I white. Initially, we have $|I|$ black-blue components. At each iteration, we can find a white node that is adjacent to the most black-blue components and color it blue as shown Algorithm 1. The set of blue nodes, i.e., set B , interconnects all black nodes in I .

Algorithm 1 Second Phase

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 $B = \emptyset$ 
for  $j = K$  to 2 do
  while There exists a white node  $v$  adjacent to at least  $j$  black nodes in different black-blue components do
     $B = B \cup \{v\}$ 
  end while
end for
Return  $B$ 

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B. Theoretical Analysis

The CDS in our algorithm is a union of set I and set B . To analyze the performance ratio of our algorithm, we first show a size relationship between an MIS and a CDS. Denote OPT as

an optimal CDS and opt as the size of OPT , we have:

Lemma 3: In a DGB, the size of any maximal independent set I is upper bounded by $Kopt$.

Proof: Since I is an MIS, by Lemma 1, no node in OPT can dominate more than K nodes in I . Thus the lemma follows: $|I| \leq Kopt$. ■

Now, we compare the size of set B to opt . Recall that B is a set of blue nodes. In other words, B is a set of all the Steiner nodes. Let T^* be an optimal tree when connecting a given set I and $C(T^*)$ is the number of the Steiner nodes in T^* , we have this following lemma:

Lemma 4: The size of B obtained from our second phase is at most $(2 + \ln K)C(T^*)$.

Proof: Let $n = |I|$ and $p = |B|$. If $n = 1$, then the lemma is trivial. Assume that $n \geq 2$, thus $C(T^*) \geq 1$. Let $v_j, j = 1 \dots p$ be the blue nodes in the order of appearance in the second phase. Let a_i be the number of the black-blue components after v_1, \dots, v_i turns blue. Since every black-blue component contains a black node which is adjacent to a Steiner node of T^* , there exists v_i which is adjacent to at least $\frac{a_i}{C(T^*)}$. Thus we have:

$$a_{i+1} \leq a_i - \frac{a_i}{C(T^*)} + 1$$

Hence we have this following recurrence:

$$\begin{aligned} a_i &\leq a_{i-1} - \frac{a_{i-1}}{C(T^*)} + 1 \\ &\leq a_{i-1} \left(1 - \frac{1}{C(T^*)}\right) + 1 \\ &\leq a_{i-2} \left(1 - \frac{1}{C(T^*)}\right)^2 + \left(1 - \frac{1}{C(T^*)}\right) + 1 \\ &\leq \dots \\ &\leq a_0 \left(1 - \frac{1}{C(T^*)}\right)^i + \sum_{j=0}^{i-1} \left(1 - \frac{1}{C(T^*)}\right)^j \\ &\leq a_0 \left(1 - \frac{1}{C(T^*)}\right)^i + C(T^*) \end{aligned}$$

For the last step in the above recurrence, we note that the second term $\sum_{j=0}^{i-1} \left(1 - \frac{1}{C(T^*)}\right)^j$ is the geometric series and it will converge to $C(T^*)$. After $i = C(T^*) \ln \frac{a_0}{C(T^*)}$ iterations, the number of black-blue components will be:

$$\begin{aligned} a_i &\leq a_0 \left(1 - \frac{1}{C(T^*)}\right)^i + C(T^*) \\ &\leq e^{-\frac{i}{C(T^*)}} + C(T^*) \\ &\leq 2C(T^*) \end{aligned}$$

Therefore, the total number of blue nodes is bounded as follows:

$$|B| \leq i + 2C(T^*) \leq C(T^*) \left(\ln \frac{a_0}{C(T^*)} + 2\right)$$

$$|B| \leq C(T^*) \left(\ln \frac{n}{C(T^*)} + 2\right) \leq (2 + \ln K)C(T^*)$$

■

Theorem 1: The approximation ratio of our proposed algorithm is $(K + 2 + \ln K)opt$ where $K = 5$ if $k = \frac{r_{max}}{r_{min}} = 1$, otherwise $K = 6(3\lceil \log_2 k \rceil + 2)$

Proof: : This results directly from Lemma 3 and Lemma 4:

$$\begin{aligned} |CDS| &= |I| + |B| \\ &\leq (K + 2 + \ln K)opt \end{aligned}$$

■

In practice, we expect that k is very small since the transmission ranges of all nodes in a network should be slightly different.

Corollary 1: If the maximum and minimum transmission ranges are bounded, then our algorithm has an approximation factor of $O(1)$.

III. DISCUSSION

In this paper, we have studied the Connected Dominating Set (CDS) problem in Disk Graphs with only Bidirectional links (DGB). The disk graphs can be used to model wireless ad hoc networks where the nodes have different transmission ranges. We have proposed an approximation algorithm and shown that the obtained CDS is within a constant factor of the optimal CDS. In particular, the algorithm first finds a Maximal Independent Set (MIS) and then connect all nodes in the MIS using the Steiner tree. This algorithm can be implemented as a distributed algorithm which we cannot show here due to the space limitation.

When nodes in a network have different transmission ranges, a node u can communicate directly to node v but node v might not be able to communicate directly back to u . It is our interest to study the CDS problem in general disk graphs where both unidirectional and bidirectional links exist. One simple way is to find a dominating set and then use a directed Steiner nodes algorithm to connect them. Note that the CDS in this case is directed. Hence we need to find a strongly connected CDS to help the routing.

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