The Complex Wave Representation (CWR) of Shape

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Over the past three decades, shape analysis has borrowed numerous formalisms, methodologies and techniques from classical physics. Curiously, there has been very little interest in adapting approaches from quantum mechanics. This is despite the fact that linear Schrödinger equations are the quantum counterpart to nonlinear Hamilton-Jacobi equations [1] and the knowledge that the quantum approaches the classical as Planck’s constant \( \hbar \) tends to zero.

Distance transforms are popular shape representations. The distance transform scalar field in two dimensions \( S(x, y) \) satisfies the static, nonlinear Hamilton-Jacobi equation \( \|
abla S(x, y)\| = 1 \). A peculiar fact is that the nonlinear Hamilton-Jacobi equation can be embedded in a linear Schrödinger equation \( -\hbar^2 \nabla^2 \psi(x, y) = \psi(x, y) \). The wave function \( \psi(x, y) \) is the complex wave representation (CWR) of the distance transform. Now, take the Fourier transform of the CWR \( \psi(x, y) \) to get \( \Psi_h(u, v) \). We see in Figure 1 (the center right figure) that the Fourier transform values lie mainly on a circle and we have observed that this behavior tightens as \( \hbar \to 0 \). The stationary phase approximation [2] can be invoked to explain the reason why the Fourier transform \( \Psi_h(u, v) \) takes values mainly on the unit circle.

\[
\lim_{\tau \to 0} \lim_{\hbar \to 0} \int_{\theta}^{\theta + \tau} |\Psi_h(u(\theta), v(\theta))|^2 d\theta = p(\theta)
\]

where \( p(\theta) \) is the density function of the unit vector distance transform gradients and \( \Psi_h(u(\theta), v(\theta)) \) is the Fourier transform of the CWR \( \psi(x, y) \) evaluated on the unit circle (in the spatial frequency domain). We see that spatial frequencies are essentially gradient histogram bins.

The unusual connection between the Fourier transform of the CWR \( \psi(x, y) \) and the distance transform gradient density opens up a new front for shape analysis. For instance, it would be interesting to examine the shape statistics (mean, covariance etc.) of the CWRs. This represents a fruitful avenue for future research.

References
