1. Let $\phi(v)=\dfrac{1}{1+e^{-av}}$.

   Show that $\dfrac{d\phi}{dv}=a\phi(v)\left[1-\phi(v)\right]$.

$$\frac{d\phi}{\mathrm{dv}}=\frac{-1}{(1+e^{-av})^2}\cdot e^{-av}\cdot a=\frac{a\,e^{-av}}{1+e^{-av2}}=a\cdot\frac{1}{1+e^{-av}}\cdot\frac{1+e^{-av}-1}{1+e^{-av}}=a\cdot\frac{1}{1+e^{-av}}\cdot\left(1-\frac{1}{1+e^{-av}}\right)$$
$$=a\ \phi(v)(1-\ \phi(v))$$

What is the value of this expression at the origin?

$$\phi(v)=\frac{1}{1+e^{-av}}=\frac{1}{1+e^{0}}=\frac{1}{1+1}=1/2$$

In the limit as *a* approaches infinity, what happens to this function?

$$\lim_{x\to\infty}\ \phi(v)=\frac{1}{1+e^{-\infty v}}=\begin{cases}1\,\mathit{if}\ v>0\\ 0\,\mathit{if}\ v<0\end{cases}.$$

2. Let $\phi(v)=\tanh\left(\dfrac{av}{2}\right)$.

   Show that $\dfrac{d\phi}{dv}=\dfrac{a}{2}\left[1-\ \phi^2(v)\right]$.|

$$\phi(v)=\frac{e^{av}-1}{e^{av}+1}$$
$$d\phi\frac{(v)}{dv}=-a\,e^{av}\frac{e^{av}-1}{(e^{av}+1)^2}+\frac{a\,e^{av}}{e^{av}+1}$$
$$=\frac{(-a\,e^{2\mathrm{av}}+a\,e^{av})+a\,e^{av}(e^{av}+1)}{(e^{av}+1)^2}$$
$$=\frac{-a\,e^{2\mathrm{av}}+a\,e^{av}+a\,e^{2\mathrm{av}}+a\,e^{av}}{(e^{av}+1)^2}$$
$$=\frac{2\,a\,e^{av}}{(e^{av}+1)^2}$$
$$=\frac{a\,4\,e^{av}}{2(e^{av}+1)^2}$$
$$=\frac{a}{2}\frac{(e^{2\mathrm{av}}+2\,e^{av}+1)-(e^{2\mathrm{av}}-2\,e^{av}+1)}{(e^{av}+1)^2}$$
$$=\frac{a}{2}\frac{(e^{av}+1)^2-(e^{av}-1)^2}{(e^{av}+1)^2}$$
$$=\frac{a}{2}\left(1-\frac{(e^{av}-1)^2}{(e^{av}+1)^2}\right)=\frac{a}{2}(1-\varphi(v)^2).$$

What is the value of this expression at the origin?

$$\phi(0)=\frac{e^0-1}{e^0+1}=0/2=0 \ .$$

In the limit as *a* approaches infinity, what happens to this function?

If *v* is positive, we must use L'Hopital's rule:

$$\lim_{a\to\infty}\ \phi(v)=\frac{e^{av}-1}{e^{av}+1}=\lim_{a\to\infty}\frac{a e^{av}}{a e^{av}}=1 \ .$$

If, on the other hand, *v* is negative,

$$\lim_{a\to\infty}\ \phi(v)=\frac{e^{av}-1}{e^{av}+1}=-1/1=-1 \ .$$

3. Let $\phi(v)=\dfrac{v}{\sqrt{1+v^2}}$ .

   Show that $\dfrac{d\phi}{dv}=\dfrac{\phi^3(v)}{v^3}$ .

$$\frac{d\phi}{dv}\text{¿}=\frac{-v^2}{(1+v^2)^{3/2}}+\frac{1}{(1+v^2)^{1/2}}$$
$$=\frac{-v^2}{(1+v^2)^{3/2}}+\frac{(1+v^2)}{(1+v^2)^{3/2}}$$
$$=\frac{1}{(1+v^2)^{3/2}}$$
$$=\frac{v^3}{v^3(1+v^2)^{3/2}}$$
$$=\frac{1}{v^3}\varphi(v)^3 \ .$$

4. Which of the functions in problems 1 through 3 can be treated as a cumulative probability distribution function. Why?

   A cumulative probability distribution function is right-continuous, monotone non-decreasing, and its limits at plus and minus infinity are 1 and 0 respectively.

   This applies to the logistic function of problem 1, however, this does not apply to the sigmoids presented in problems 2 and 3 since they take negative values on negative arguments.

5. Show the binomial expansion of $\dfrac{1}{(1-wz^{-1})}=\displaystyle\sum_{i=0}^{\infty} w^i z^{-i}$ .

$$\frac{1}{(1-wz^{-1})}=\sum_{i=0}^{\infty} w^i z^{-i}\Rightarrow$$

$$
\begin{aligned}
1 &= (1 - w z^{-1}) \sum_{i=0}^{\infty} w^i z^{-i} \\
&= \sum_{i=0}^{\infty} w^i z^{-i} - (w z^{-1}) \sum_{i=0}^{\infty} w^i z^{-i} \\
&= \sum_{i=0}^{\infty} w^i z^{-i} - \sum_{i=1}^{\infty} w^i z \\
&= w^0 z^0 \\
&= 1
\end{aligned}
$$

*Q.E.D.*