Nearly linear time algorithms for graph partitioning, graph sparsification, and solving linear systems

*Abstract*
We design preconditioners for symmetric diagonally-dominant linear systems that enable their solution in time nearly-linear in their number of non-zero entries. Our algorithm for constructing the preconditioners makes use of two novel algorithmic tools. The first is a nearly-linear time algorithm that takes as input any weighted graph \( A \) and outputs a weighted graph \( B \) with at most \( O(n \log^{O(1)} n) \) edges such that the Laplacian matrices of these graphs, \( L_A \) and \( L_B \), satisfy

\[
\forall x, \quad x^T L_B x \leq x^T L_A x \leq (1+\epsilon) x^T L_B x,
\]

for any \( \epsilon > 0 \). In turn, this graph \( B \) is obtained from a fast algorithm for the following approximation version of the graph partitioning problem: on input \( \phi \) and a graph \( G \), output a cut of isoperimetric number at most \( O(\phi^{1/3} \log^{O(1)} n) \) separating at least \( 2/3 \) as many nodes as the best cut of isoperimetric number \( \phi \). That is, it attempts to find the most balanced cut of isoperimetric number close to \( \phi \).