INTRODUCTION TO THE AESTHETIC COMPUTING METHOD FOR TEACHING ALGEBRA IN MIDDLE AND HIGH SCHOOL

Paul Fishwick, PhD
http://www.cise.ufl.edu/~fishwick/aescomputing

Draft Date: 9/28/2005
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction to Aesthetic Computing for Algebra</td>
<td>3</td>
</tr>
<tr>
<td>Middle School Lesson Plans, Handouts, Teacher Resources</td>
<td>10</td>
</tr>
<tr>
<td>High School Lesson Plans, Handouts, Teacher Resources</td>
<td>24</td>
</tr>
<tr>
<td>Software Tools (2D and 3D)</td>
<td>41</td>
</tr>
</tbody>
</table>
Motivation

Modern media shape the way we view the world. With a natural language such as English, we have seen a gradual evolution of representation from voice to handwriting, print, photography, radio, television, cinema, and video game consoles. This only briefly covers the possibilities of media, since there is active research in holography, 3D displays, and new areas on the horizon such as pervasive and tangible computing. The same semantic “content” is represented in different ways. This evolution is not based on replacement. Television does not eliminate the radio, nor do video games eliminate the book. Instead of replacement, the world of media create an augmentation—new ways of seeing and hearing similar content. We see the same content through multiple representations.

Considering this history of English, and contrasting this with formal languages such as mathematics, we find a different kind of evolution—one based on improving the richness of mathematics without significantly affecting representation. Why is it that we cannot also have an evolution, or perhaps a revolution, in the ways in which we might view and hear mathematics? To some extent, this evolution is taking place in what we might call the solution space of mathematics: plotting graphs, solving equations, running programs. In this solution space, new technologies help us to visualize geometric proofs and equation plots in new ways. The problem is that there has been little emphasis on the problem space. When we express and manipulate mathematical expressions, we are dealing with the problem space. Can the problem space also be a place where we might create new representations using our latest technologies? Can we create game-like, highly visually appealing, worlds to attract students to mathematics? Can mathematical expressions look like a video game, feel like a physical sculpture, or be read like a story? This is the emphasis on aesthetic computing, and the corresponding method described in this document.

The method provides the following definite benefits:

- **Order of operations:** students learn the order of operations when constructing the expression trees. This provides the student with a clearer understanding of the structure of algebraic expressions.

- **Interdisciplinary activity:** students when making expression trees, and the 2D, 3D, or physical objects that they represent are forced to think in terms of design, art, and English sentence structure (i.e., for the story type of target object). Thus, mathematics is linked closer to the literary and fine arts. This provides a way for math teachers and other teachers to relate their work to each other.

- **Introduction to computing:** the exercise involving expression trees has a close link with learning how to use/program a calculator or computer since mathematical expressions need to be made explicit, and the expression tree can be used as a means of computation or calculation to obtain results.

The method has these perceived, untested, benefits which may eventually become hypotheses for testing: (a) improved memory retention for complex expressions—the
mnemonic effect of using icons; (b) improved motivation for doing algebra; and (c) improved grades in math class.

This document is divided into four parts:

1. **Introduction** – what you are reading now, to create an overview of the aesthetic computing method
2. **Middle School Track** – lessons and handouts for Middle School: pre-algebra or algebra (courses: Math 6,7,8)
3. **High School Track** – lessons and handouts for High School: algebra and integrated math (course: Integrated Math 3)
4. **Tools** – 2D and 3D tools (virtual algebra machine) to assist students and teachers

If you see a small icon that looks like Rodin’s *The Thinker* (the figure on the left), you can probably skip this extra material unless you want more detail. When we consider problem space versus solution space, these two phrases relate in ways similar to *syntax versus semantics* (the structure of a thing versus its meaning, which is often the interpretation, or *behavior*, of the syntax). You can also think of it from a computing perspective: an equation is fed into a machine, which then operates on it, producing output in the form of a plot, a re-arranged symbolic expression, or something else. Thus, mathematics has focused on novel representations for what comes out of the machine, but not what goes into it.

### 1. Goal

The goal of these lessons, handouts, and implementation should be to provide a creative exercise in teaching the subject of algebra that instills student interest in the subject. The purpose is to help motivate the student in algebra by allowing them to be creative. The most basic assessment strategy is a video-taping of the class, surveying of students after the lessons are complete, and teacher interview as to the success or failure of the method. The documentation is composed of this introduction to the aesthetic computing method, followed by lesson plans, handouts, and assessment mechanisms. There are two tracks: MIDDLE SCHOOL and HIGH SCHOOL. We recommend that teachers look at both tracks and take examples that are relevant to their classes.

The overall strategy is for the student to begin with a *source* algebraic expression in text and for this source expression to be represented by a target product. The *target* may be something dynamic such as music, animation, or a story or something physical and concrete such as a sculpture, architectural layout, or form of theatre.

### 2. Method

The aesthetic computing method is composed of the following steps. Some of these steps may take different amounts of time to teach, depending on the grade, and some might be merged with other step to save time.
STEP 1: Standard Notation – this is where we begin, with traditional (standard) textual algebraic notation for one or a set of algebraic expressions. The “grammar” of traditional notation is discussed along with the implicit operations by virtue of their relative position in the expression. For example multiplication are sometimes expressed implicitly as “ab” instead of “a*b”, and powers as “a^2” instead of “a²” or “a*a”.

STEP 2: Prefix Notation – the standard algebraic expression is modified into a prefix expression. This is still in text form, but the operators are located before the operands. For example, a(b+c) becomes a * (b + c) to first make all operations explicit, and then the prefix notation is * (a, + (b,c)). In “computer parlance”, this is also known as “functional notation” since all operators (i.e., functions) are specified first and grouped with their arguments (i.e., operands). The transformation from standard to prefix notation is easiest in gradual steps: (1) a(b+c), (2) *(a,b+c), (3) *(a,+(b,c))

STEP 3: Expression Tree – we take the textual prefix expression and convert it into an expression tree with the root of the tree at the top and the leaves at the bottom. Thus, this tree is inverted, or flipped, from what a tree normally looks like. A tree is composed of nodes and branches, with external nodes (i.e., “leaves”) and internal nodes. The expression tree is created from the prefix notation expression by reading it from left to right, and building the expression tree in a top-down manner from root to the leaves. Operators appear as internal nodes, while variables and numbers are leaves. For example, reading the prefix expression “*(a,+(b,c))” from left to right gives us a root tree node of “*” with a left branch ending with a leaf “a” and a right branch ending with an expression “+(b,c)”. This expression is then expanded further to be a node of “+” with two leaves “b” and “c”.

The creation of the expression tree reinforces the order of operations. The order of operations proceeds from the bottom-up, or from the leaves of the tree up to its root.

STEP 4: Concept Mapping Key – the first stage in mapping is to develop a key by taking each element of the expression tree and mapping it to a corresponding English word or phrase. The key is like the key of a chart or map with two columns: the left column containing the element (branch and node) of the tree and the right column containing the target English representation.

STEP 5: Concept Map – the mapping associations (left and right columns) inside the key created in the previous step are applied to yield a modified expression tree which now has English labels for both the tree nodes and branches. Each branch of the tree should be represented with an arrow rather than a line if there is the possibility of ambiguity. The arrow is useful in capturing the node relationship as with “a is-plugged-into b”, rather than “b is-plugged-into a”, which has a different meaning. We call this resulting tree, with arrows for branches, a concept map. This map is a visually represented tree with node and branches labeled with English words or phrases.
STEP 6: **Design Mapping Key** – the second mapping stage involves creating a set of mappings, in the form of another key that maps the English word/phrase concept to a 2D, 3D, or physical element.

STEP 7: **Design Map** – The design mapping key is used to create a new tree that is labeled with visual icons present from the previous step.

STEP 8: **Representation** – The design map is “realized” into a 2D, 3D, or physical representation using the design elements in the prior step. This realization may take several forms: story, structure, or space for example.

This description of the aesthetic computing method is one that is made specific to teaching algebra. More generally, the method is slightly altered: Step 1 involves textual notation for a mathematical structure which may be a data or program structure, in addition to an expression. Step 3 is generalized to a graph, rather than a tree.

### 3. Mapping Strategies

Steps 4 through 8 in Sec. 2 require creative mapping approaches. There are some guideline templates that will help this process. They are based on whether we consider sample domains, tree mapping as a whole, or whether we are simply looking at potential mapping possibilities for individual tree nodes and branches.

#### 3.1 Domain, Style and Case Study

The first strategy might be to consider a specific application theme or domain of interest. For example, a domain might be theatre, cinema, architecture, or sculpture. Within each domain, there are styles and case studies. For cinema, we might have fantasy as a style, and then *Harry Potter* and *Lord of the Rings* as specific case studies for the fantasy style. For architecture, we might have a modern style with Bauhaus or Gaudi serving as the targeted case studies. In some cases, the teacher may wish to constrain the assignment to a given domain, style or case. Or, students could be free to choose.

#### 3.2 Category

The categories *story*, *structure*, and *space* serve as useful archetypes to help constrain and guide mapping.

**Story**: operations will be internal nodes in the tree diagram, and will usually map to verb phrases with other nodes being nouns. The branches would be adverbs or left unlabeled.

**Structure**: all tree nodes will be physical objects, and branches will either be connecting objects (wires, tubes, cables) spatial relations (i.e. adjacent-to, below, next-to, on-top-of etc), or containment relations (i.e. inside-of, outside-of).
**Space:** all tree nodes will be spaces, and branches will be connecting or containment relations as in Structure.

The story is the only type that involves the flow of time and it must be made clear as to which nodes occur before other nodes (i.e., node order) to “tell the story.” For example, one might say that the story begins at the leaf nodes of the tree and moves upward with the passage of time. Explicitly labeling nodes or branches with numbers (i.e., 1,2,3 …) indicates the temporal order of events in the story. Separate branches, in such an approach, may designate parallel story lines. For the story, the student should number each node to tell in what *order the tree is traversed.*

Semantics that are used to label the tree can ultimately result in representations that span a wide variety of human interfaces. For example, it is possible to map music or sound instead of, or in addition to, existing relations—thus obtaining hybrid audiovisual representations. Moreover, different media can be used in the representation – physical or computer generated.

### 3.3 Relation

If we look at the tree structure itself with nodes and branches, we are led to two sorts of relations between two nodes: temporal and spatial. Nodes generally *map to physical objects.* A branch connects two tree nodes, and the semantics for this branch depends on one of two cases:

- If the branch represents a temporal relation between two nodes then we will use a *verb phrase* for the branch semantics, and the resulting tree will be something dynamic such as a story or animation with the flow of time being explicit.
- If the branch represents a spatial relation between two objects then this relation can be one of these types: proximity, connection, containment, which tend to indicate *prepositional phrases.* Proximity relations suggest relative position and include phrases such as next-to, above, below, in-front-of. Connections are physical objects that connect two objects such as a wire, tube, or pathway. Connections are phrases with object arguments such as connected-to(wire3) meaning connected to the object “wire3”. Containment refers to encapsulation of one object in another as in a Venn diagram.

### 4. Guidelines

**4.1 Completeness:** *The mapping from source to target should be complete:* all elements in the source should be represented in the target.

There are two parts to this: completeness in *syntax* and in *semantics.* The syntax of an expression is its denotational quality referring to its grammatical structure. The source is an expression that contains many elements such as numbers, relations, variables, and operators. In any representation of the source expression, we should try to ensure that each of these elements has a counterpart in the target representation. This is so that we do
not miss out on mapping an element. This syntactical aspect of completeness is the most important. The other part is semantic mapping – we’d like to ensure that we map any semantics in the source to the target. For example, if “x” represents a variable in the source, then we need a corresponding thing in the target that represents a variable even if it looks very different than “x”.

Semantic mapping can become complicated. For example, there are many expressions that have “implicit semantics”. For example, representing $Y = aX + bZ$ has no additional semantics—it is an abstract expression and will likely not remind the student of anything in particular other than being an expression containing variables, a “+” operator and a relation “=”. However $Y = aX + b$ is a “taught template” for the equation for a line. If the student has been taught this previously, then the pre-existing knowledge of “line” represents information that may need to be carried forth to the target in some way.

4.2 Consistency: The mapping from source to target should be consistent: similar elements in the source should map to similar elements in the target.

There are two types of consistency, based on syntax and semantics. For syntax, we must ensure that if we represent an “x” as a grey stone in the target, we must ensure to represent all instances of “x” with grey stones. This can be thought of as consistency with respect to instances.

Regarding consistency in categories (or types), in a source expression “x” and “y” are both types of “Variable”, which is the category. One would then expect that target representations of “x” and “y” should also be related (i.e., fall within the same category). It is not yet clear how stringent we should be on these categories. For example, if “x” and “y” map to a red stone and a green stone, clearly the category of “stone” is at a fairly low level, consistent with “variable”. However, if we decided to represent “x” and “y” as a pink bear and a blue lion, the common category of “animal” is at a fairly high level in comparison.

Another aspect of this rule to consider deals with the visual or structural complexity of the objects used in the target representation. A stone is fairly simple in structure, but a lion has a lot of structure and attributes with pre-conceived student notions of what a lion looks like, or the way it behaves. The potential trap is that a student may become confused with the complexity of a target, imagining that certain object attributes (i.e., the size or color of the lion) has some meaning when reflecting, say, the magnitude of the variable “X”. The question is: what target object attributes do we ignore during sign formation? The mapping chosen may be singular to one specific algebraic expression, or it may provide scalability. This suggests a rule in mapping that hints that target representations that are scalable (when scaling the source), may represent a more powerful means for representation. Scaling is difficult, however, for a story mapping, since stories will tend to work well for that one algebraic expression, and then fail if other variables or operators are added. Structures and spaces appear to have greater scalability potential.
5. Assessment Methods

5.1 Teacher to student
In assessing the student’s representation work, you may wish to consider the following:

- The completeness of the representation (Sec. 4.1)
  - Are the keys one-to-one in terms of mapping?
  - Can you go “in reverse” from the target to the source, given a key?
- The consistency of the representation (Sec. 4.2)
  - Take similar or identical elements in the target – are they similar or identical in the source?
- The aesthetic quality of the representation:
  - Are the target elements clearly perceived (visually or aurally)?
  - Does it look like the student spent considerable time on the product?
  - Does the target product have an appeal?
  - Does the target product represent a strong degree of creativity?

5.2 Student to student
It might be useful for students to grade, or otherwise judge, others’ work in a group assessment.

5.3 Teacher to Curriculum
What are the teacher’s views of the approach – strengths and weaknesses?

5.4 Student to Curriculum
What are the student’s views of the approach – strengths and weaknesses?

6. Acknowledgments
Micah Branum created the figure on page 1. The field of aesthetic computing has been forged through the activities of numerous University of Florida students. During the Summer of 2005, two teachers, Jodee Alice Rose and Katrina (Katie) Indarawis of Gainesville, Florida have been instrumental in putting together the attached two tracks, with Jodee being chiefly responsible for the Middle School track and Katie for the High School track. Both Jodee and Katie are recipients of the National Science Foundation Research Experiences for Teachers (RET) award given for the Summer 2005 session.

The work performed by the author (Paul Fishwick) and teachers were funded in part by the National Science Foundation under grant EIA-0119532 entitled “An Investigation into Aesthetic Computing within the Digital Arts and Sciences Curricula” with co-PIs Timothy Davis and Jane Douglas. There are many students who have in the past contributed to the area of aesthetic computing. The students actively involved in aesthetic computing at this time are Salam Daher, Hyunju Shim, Zachary Ezzell, Yuna Park, and Chen-Yuan Huang. Minho Park finished the PhD and recently accepted a position as Assistant Professor at Stephen F. Austin University in Texas. Zachary Ezzell is an Undergraduate Engineering Scholar, 6 month appointment.
middle school

The purpose of this level is to encourage creativity in representation, and to spend additional time on the steps from the original expression to the final product.
Lesson One: Changing Mathematic Notation to Prefix Notation (2 days)

Objectives:
• Students will realize that math problems often use a shorthand notation that needs to be expanded to gain full understanding of what is being asked of the student.
• Students will learn to group algebraic expressions by a main operator

Activities:

Objective One
• As a class, lead by teacher, students will identify different types of operators.

Questions to ask:
• What is an operator? (Tells us what to do in a math problem)
• What is another word we could use for that? (Signs)
• List the different operators. (+, -, *, /, powers, square root, division bar)
• What are other ways we are told to do these operations? (2a, 2(a), 2/a, etc.)
• Explain “implicit” versus “explicit”. (Use examples from above to solicit responses from students as to which are implicit and explicit)

• Teacher will present several problems with varying complication and show the students where implicit operations are changed to explicit operations. (5 examples)

Work out the following problems by changing implicit operations to explicit:
• 6g = 6*g
• 2(a + b) = 2 * (a + b)
• x(9c) = x * (9*c)
• 14t - 3/ u = 14*t -3/u
• 2c + 4j - 2/y = 2*c + 4*j – 2/y

• Teacher will walk students through several examples (implicit to explicit) picking volunteers and random students to answer aloud. (10 examples)

9(5-y) = 9 * (5-y)  4 + 7x= 4 + 7*x
2z (5g)= 2* z * (5*g)  x² = 3= x^2 - 3
8z + 3= 8*z + 3  8(4x + 9)= 8 * (4*x + 9)
6 (9+y)= 6* (9+y)  4y + 5y + 8y= 4*y + 5*y + 8*y
7y (4/h)= 7* y *( 4 /h)  (7z + 5)(2g + 3)= (7*z +5) * (2 * g + 3)

• Students work on problem set. (WS or Overhead Set 20)

See attached problem set titled Changing Implicit Operations to Explicit Operations

Objective Two
• As a class, lead by teacher, students will discuss the order of operations.

Questions to ask:
• What is the order of operations? (PEMDAS)
• Why is the order of operations important? (It tells us what order to do a problem in when there are several different ways. Without it, answers would be different
throughout a class

• Show an example of a problem using order of operation incorrectly and then correctly
  • 4 + 10 * 2 (wrong = 14 * 2 = 28) (correct 4 + 20 = 24). The answers are different!
• What was the first thing we did in the last example? (adding)
• What two parts were we adding? (10 * 2 and 4)
• Show students an example of Prefix Notation in three steps:
  • 4 + 10 * 2
  • + (4, 10 * 2, 4)
  • + (4, * (10, 2))
• Does that make sense to anyone?
• This is a thing called “Prefix Notation” That means what you are supposed to do (the
  operation) is placed at the front of the problem. Pretty confusing, huh? Well, it
  should help you to understand what is going on in a problem a little better.

• Teacher will present several problems with varying complication and show the students how
to group operations and operators in prefix notation. (5 examples)

As with most algebraic work, it will help the students to go through the three steps for each
problem to avoid confusion. The answers below only represent the final answer. Notice that the
students may write the words or use the symbols.

• 6g = *(6, g)
• 2(a + b) = *(2, + (a, b))
• x(9c) = *(x, *(9, c))
• 14t - 3/u = - (* (14, t), / (3, u))
• 2c + 4j - 2/y = - (+ (* (2, c), * (4, j)), / (2, y)) wow! That was hard work!

• Teacher will walk students through several examples (implicit to explicit) picking volunteers and random
students to answer aloud. (10 examples)

• 9(5 - y) = *(9, - (5, y))
• 2z (5g) = *(2, z), *(5, g)
• 8z + 3 = + (* (8, z), 3)
• 6y (9 + z) = * (6, + (9, y))
• 4 + 7x = *(4, + (7, x))
• x² - 3 = - (^ (x, 2), 3) or - (*(x, x), 3)
• 8(4x + 9) = *(8, + (*)(4, x), 9)
• 4y + 5y + 8y = *(4, y), *(5, y), *(8, y)
• 7y (4h) = *(7, y), *(4, h)
• (7z + 5)(2g + 3) = *(7, z), *(5, h), +(2, g), 3

• Students work on problem set independently. (WS or Overhead Set, 20 examples)
Use worksheet attached titled Implicit Operations to Explicit Operations. Answers attached.

Wrap Up Questions:

How many of you felt that this method was interesting?

...Helpful?

...Really, really hard?
Lesson Two: Expression Trees

Objectives:
- Identify operators in given math equation
- Demonstrate knowledge of how the order of operations will affect the order of branches on a tree
- Using previous lesson (prefix notation), students will arrange various algebraic structures into expression trees

New Vocabulary:
Define expression tree (branch, leaf, and root)

Activities:
Objective One was covered in the last lesson. Teacher should give students the following quick quiz to test that objective has been met.

Identify the Operators in the following problems:
- \((7z + 5)(2g + 3)\) = \((7z + 5) * (2g + 3)\)
- \(x^3 + 6k\) = \((x \cdot x \cdot x) + 6k\)
- \(9(5-y)\) = \(9 \cdot (5-y)\)
- \(y/3(5)\) = \(y/3 \cdot 5\)
- \(14t - 3/u\) = \(14 \cdot t - 3/u\)
- \(5 - 3y \cdot 2\) = \(5 - 3 \cdot y \cdot 2\)
- \(q(\text{u i z})\) = \(q \cdot u \cdot i \cdot z\)

Review answers with students.

Objective Two and Three:
- Review lesson one's results briefly (Remember how we grouped the problems by operators? It was easy for some and more difficult for others. We will be doing something like that today, but a little different, and more fun)

Questions:
- What was an operator? (+, -, *, /, etc.)
- How did we figure out which operator to put out front in the sets we did? (Whichever operation was last in PEMDAS.)
- How many of you have heard of trees? (All will) and in math? (Tree diagrams are used for prime factorization and branch and leaf diagrams are somewhat like trees)

Introduction: "Today, we are going to be making math problems into 'trees'. What are the parts of a tree? (Wait time) Well, our trees will have branches, leaves, and roots. I will show you two examples and then we will do several together and then you will work on some on your own. We are going to use the problems we have already worked out so that we have even less work to do! Please take out your Lesson One worksheet."

Objective Three:
Using the handouts for the previous two assignments, student should put their prefix notation problems into tree diagrams. Teacher will lead in two examples and then solicit student

---

1 Teachers may wish to choose whether to use “/” or “÷” for “division”, or both.
help for 4 of the previously worked out examples (see lesson one) and the teacher will assign student to do 1-8 on the lesson one worksheet.

Assignment:
Work out the following problems FOR the students.

\[
y = mx + b \rightarrow y = m \times x + b \rightarrow (y, +(*(m,x), b))
\]

\[
4z - 6 \rightarrow 4 \times z - 6 \rightarrow (-*(4,z), 6)
\]

1.

As you work, explain that the circles are the nodes, the lines are the branches and the leaves come off of the branches. It will help the students to see the correlation if the teacher talks through the prefix notation as a guide of what to do next.

Example of "out loud thinking" for example one:
Y equals MX plus B. Okay, so if we put that into expanded form, we have "Y equals M TIMES X plus B" okay! Now if we put that into "prefix notation" we have this: Equals at the front, because that is the last thing we have in a problem, right? So, Equals, what equals, hmm, well. Y equals something so, Y. And then what else? Hmm. Oh, What kind of problem is on the other side of the equals sign? Well, it is an addition problem. We are adding a multiplication problem and the variable B. Okay, so what should go out front? It's a fancy addition problem so, + should go out front, okay so now we have "=(y, +..." what now...? Oh okay, the multiplication of m and x. that looks like "(m, x) and then we are adding b to it, so it is + (+(*(m, x), b) okay, whew, got that, now how do we put it into an expression tree?

Equal came first so we put that in the top ROOT (It's just a circle, don't get nervous). Then we will have two branchess coming from the root. Why two? (There were two sets in the prefix notation). On the left one, I put a Y underneath. Notice I didn't put a circle, you don't have to. Now on the right, at the bottom of that branch I am going to put another root and in it I am going to put a plus. How many things was I adding in the problem? Two really; a multiplication problem and B. So I have two stems coming off my plus root. Got it, almost there! On the left branch I am going to do ANOTHER root. Remember, that is a circle. Inside it, I will put a multiplication sign. Then on the right branch I will put a B. Are we done? Nope, not quite yet. We need to put our last branches and leaves on the tree. How many things were being multiplied? Two? So two stems and then leaves of m and x. Now we are done.

Work out the following problems WITH the students

\[
3. \quad 6 + (7, x)
\]

\[
4. \quad y^2 + 7
\]

\[
5. \quad 4x + 2
\]

\[
6. \quad \frac{bh}{2}
\]

After working put problems with students, assign #1-8 on lesson one worksheet as class work/homework. Check in class on a board/overhead the following day.

Wrap up questions:
Did anyone think that making expression trees was really easy once you had the Prefix Notation in front of you?
Do you think you could solve these problems more easily now that you have seen their trees?
Lesson Three: Concept Mapping

Objective
- Using previous two lessons, the students will learn to associate trees and their connections with English meanings to make the trees more interesting to the students.
- Students will utilize familiar expression trees and attach English semantics, as discussed in class to make logical connections on their trees.

Words to know:
- Proximity
- Containment
- Connectivity

Introduction: “Today we are going to do something that won’t seem like math at all. We are going to use all of the lessons we have done so far with aesthetic computing to accomplish it though. Please take out your expression tree homework.”

Have completed expression trees available for students who were absent or don’t have their homework.

Activities:
- While students are taking out materials, pass Concept Mapping Rule and Guideline Sheet to each student. Read over instructions for concept mappings as a class.
- Present the students with an example of a concept map.

Try a concept map with students. You may choose to use the map given as a guide and may prompt students with the following theme.
Structure – Candy Mobile

Using $a^2 + b^2 = c^2$

Ask:
- What roots must be the same? Which leaves?
- What candy could we use as connectors? As roots?
- Do you think you could build something like this?

Label each root, leaf and branch as the class comes up with it. Make sure that it is categorically consistent and follows all of the rules on the sheet.

Class work:
Have students choose and copy three of their completed expression trees and try one of each type of concept map (story, structure, space). Remind the students that they must actually write the English words on the expression tree. Like in an English class, the students should be encouraged to brainstorm before they start their final concept maps.

Inform the students that they will be choosing one of these concept maps for a final project during the next lesson.

Wrap up Questions:
How many of you knew that you could connect English or Art or Architecture to math?
How many of you enjoyed changing trees into concept maps?
What other rules or tips could you add to the rule page to make concept maps easier to do?
RULE 1: The mapping from source to target should be **complete** – that is all elements in the source (numbers, relations, variables, operations) should be represented in the target.

This means everything in the problem needs to be represented by SOMETHING in the final product.

RULE 2: The mapping from source to target should be categorically consistent.

Categorically consistent? It means that operators should all be in the same category (or family) and the same with numbers and variables and relations. For example, saying x is cheese and y is a gopher is not categorically consistent (no matter how much you like cheese and gophers!).

**Guidelines:**

1. First you must create an expression tree from your source (source here means problem). It is best to put your source into prefix notation as this makes it MUCH easier, and your teacher will probably require it.
2. Then take the basic expression tree and perform a semantic tagging procedure, which means that each root, leaf and branch in the tree must be clearly labeled with an English phrase.
3. Next, make sure to make a Mapping Key. It is important to make a key so anyone who looks at your work can know which word relates to what part of the expression tree. This also helps because sometimes when creating a concept map, your words may get all scrunched up. This way you can keep your thoughts clear and neat.
4. Choose from three different types of representations:
   a. **Story**-- Here you will label the roots with verbs and the leaves with nouns. The branches will be adverbs or indicate the passage of time (next, after, then, etc.). Time will usually go from the leaves up. You should number the story sequence on your concept map. Remember, two things CAN happen at the same time (if the leaves are on the same level this might apply to you!).
   b. **Structure**-- Here all the roots and leaves will be physical items. The branches will connect the objects, contain them, or tell how close they are to one another (ex., tube, wire, inside, around, on top of, next to, etc.). Your final product will look like a sculpture with this choice.
   c. **Space** -- Here the roots and leaves will be rooms or areas and the branches will either connect the areas or contain them. The final product will look like a place (ex. Dorm, garden, spaceship, etc.).
Lesson Four: Representation

Objective:
- The students will be asked to make a final representation of one of their previously tagged concept maps.

*The actual representation maps the tagged tree diagram to a target object. That object may be in various forms: (a) a drawing or painting, (b) a 2D or 3D computer model, or (c) a physical model made of a material such as paper, cardboard, clay, or wood.*

Introduction: “Remember, last lesson I told you that we would be using those concept maps for a special project. Well this is it. Today is the day. Today, we will begin the final stage of this unit. I will be asking you to actually create the ideas that you started last time. What do I mean? Remember the candy mobile we talked about last time. Imagine that we actually had to make that for a grade. Do you think you could do it still? Well that is similar to what I am asking of you.

What were the three types of representations we talked about last time? (Wait time) Right, story, structure and space. So here is the assignment. You will be asked to either draw, or build your concept map. What do I mean? Let’s say I had to make that candy mobile, I could either draw (or paint) it, model (or draw) it on a computer, or go out and buy the candy and stick it all together and bring it in. That is what you will have to do. Of course, since we did a candy mobile in class, I am asking that you not copy that idea, which is why you did your own maps. I will show you some finished examples. If you would please take out your completed concept maps, we can get started.”

Activities:
- Show students examples provided of different types of representations.
- Pass out guideline sheet and set due date on the project.
- Have students write up a proposal of what type of project they would like to create.
  Posting the following questions for the students will make it easier for them to focus their ideas which the students should submit for your review.
  - What is the expression you are using?
  - What does its expression tree look like? Copy your concept map.
  - What kind of metaphor have you chosen (story, structure, or space)?
  - What is your theme (ex. skateboarding, nature, manga, food)?
  - Do all of your roots, leaves and branches match in theme? Explain how they match.
  - Have you made both a mapping key and a design key? Attach both.
  - Draw a labeled sketch of your final product.
- Have students meet with the teacher to finalize their plans.
- Collect project on due date and grade using attached rubric.
Lesson Four: Grading Rubric

Requirements:

- Students will turn in a written proposal for their project (6 pts)
  - Points based upon six questions given to the students to answer.
- Project is worked on during given class time (3 pts)
  - Three days maximum should be given in class, and points can be deducted for off-task behavior.
- Project is turned in on time (5 pts)
  - Points may be deducted for each day the project is late.
- Student uses good craftsmanship (5 pts)
  - Points can be deducted per major construction/cleanliness issue.
- All nodes are categorically consistent (8 pts)
  - Teacher may choose to give multiple points per correct node, or deduct for inconsistencies.
- Theme is apparent and consistent (5 pts)
  - Teacher may deduct per inconsistency.
- Student turns in both the mapping and design key (2 pts)
  - Teacher should deduct points for incomplete keys.
- Project conforms to proposal idea (6 pts)
  - Teacher may choose to deduct points per unnecessary change.

Total points possible = 40

Grading Scale

"A" = 40 to 36 points
"B" = 35 to 32 points
"C" = 31 to 28 points
"D" = 27 to 24 points
"F" = below 24 points
Representation Guidelines and Instruction Sheet

Due Date__________________

Using a completed concept map from our aesthetic computing lessons, you are being asked to create an art project. Your project may be 2-dimensional (drawing, painting, collage, etc.) or 3-dimensional (sculpture, mobile, model, etc.) or created on a computer in either 2 or 3 dimensions.

Use the following guidelines to achieve the best grade possible.

• Have a strong theme in mind, something that means something to you. It is always easier (and more fun) to do something if you are interested in it.
• Always make sure your categories are consistent. Even if you think they match, it is always a good idea to ask a friend, family member or other teacher to see what they think.
• Before you turn in your work, make sure that you have completely represented ALL of the roots, branches and leaves on your expression tree. Having a design key will keep you on track.
  o To make a design key simply draw or write what symbol stands for what. That is, if you have a peppermint that stands for c, draw “peppermint=c”. Do this for each part of your concept map.
• Use good craftsmanship. That means no glopping glue or paint. In other words, make it look good, like you put TONS of effort into the assignment.
• Keep all documentation for this project. All of your trees, maps, keys and questions will be due with the project on the due date. Points will be deducted from the final total for missing information.

With that said, your project is due on the date above. At that time, you must turn in a packet containing the 7 questions from lesson four WITH all the documentation the questions request (attach your key, mapping, etc.). You must also turn in your completed project at the same time. Points are deducted for late assignments. For more information, you may ask me anytime before the due date (Please make it before the last day!). You will be given a maximum of three days to work on this project in class. Beyond that, the final project is your responsibility. This should be a fairly easy and fun project so don’t let it get you!
Aesthetic Computing: Teacher Resources

Dorm Life – Example of Structure

dorm life

\[ y = mx + b \rightarrow y = m \cdot x + b \rightarrow = (y, +(* (m, x), b)) \]
Aesthetic Computing: Teacher Resources

Dorm – Example of Space

\[ y = mx + b \rightarrow y = m \cdot x + b \rightarrow (y, +(* (m, x), b)) \]
Joe is a chef at Burgeroo Fast Food Palace. He absolutely loves his job. Everyday he goes to work first thing in the morning and mixes the spices and meat together. Then after he had made all of the patties, he puts them in a frying pan and cooks them until they are well done. The next step is to prep the buns. Once Joe has cooked up all the burgers, he adds them to the buns and puts them in their special yellow wrapping. Everyone loves Joe’s burgers. In fact, you can always find a line of people outside Burgeroo Fast Food Palace who are just waiting for a bite!
The purpose of this level is to encourage creativity in representation, and to alleviate student frustration and lack of motivation for the trigonometric identities.
Lesson One: Expression Tree

Objectives:
- Students will be able to construct 'trees' given simple algebraic equations.
- Students will be able to match the correct tree to trigonometric identities.

New Vocabulary:
Define expression tree

Activities:

Objective One
- **Intro:** "Today we are going to try something new. We will start with some simple equations and turn them in to what we call 'expression trees.' The way this is going to work is that I will do two of them on the board, then we will do two together, and then you will do four of them on your own. Any questions?"
- On the board, teacher writes: \( y = mx + b \)
- **Questions:**
  - "What is the primary operation?" --Answer: "addition" – “\( mx + b \)"
  - "What are we adding together?" --Answer: “\( mx \) and \( b \)"
- "So, we can start our tree like this:

\[
= \\
/ \ \\
y \\
/ \ \\
+ \\
/ \ \\
mx \ \\
/ \ \\
b
\]

- **Question:** “Are there any other operations in this that we haven't dealt with yet?”
  (wait time) --Answer: “\( mx \) is multiplication.” “Great! So \( mx \) is multiplication and we need to expand that out in our tree.“
- On the board, teacher adds…

\[
= \\
/ \ \\
y \\
/ \ \\
+ \\
/ \ \\
* \\
/ \ \\
b \\
/ \ \\
m \ \\
/ \ \\
x
\]

- Teacher will continue in this manner for the following examples:
  Teacher does: a) \( y = mx+b \)  b) \( x^2 + y^2 = 1 \)
  Together do: c) \( xy = 1 \)  d) \( y^2 = 1 + x^2 \)
Students do:  

e) $2y = xy^2$  
f) $y/2 = x/(1+x)$  
g) $y/2 = (1-x)/z$  
h) $y = x(90-z)$

- As students are working, the teacher will be circling the room and helping students. When they are finished, teacher will go over e)-h) above (that was assigned to students).
- If the teacher decides that the students need a little more practice with these, have the students do the following:
  1. $y = b + c$  
  2. $y = a/b$  
  3. $y = 1/x^2$  
  4. $y = c \cdot (a/b)$  
  5. $y = a \cdot (b + c)$

Objective Two
- If the above five problems were assigned, the teacher will go through the answers and answer any questions as necessary. Otherwise,
- The teacher then puts the transparency of the trigonometric identities on the overhead. The students are asked to copy these down in their notes and then to observe the forms of the equations.
- “Let’s compare these identities to the forms of the equations we just did. For example, let’s look at $x^2 + y^2 = 1$. Which trig identity has the same form as this equation? (wait time) --Answer: “$\cos^2 \theta + \sin^2 \theta = 1$”
- “Now, categorize the rest of these equations by writing the correct letter beside each equation that matches the form. Don't make this harder than what it is, it's simply a matching game.”
- Teacher will walk around and help students, giving them approximately 8 minutes to match all the equations with the corresponding trig identity. Then go through the correct forms (approx. 5 min).
- **Questions:** “Does everyone see how these identities can take the form of some simple algebraic equations? Anyone having a problem with this or have questions?” (wait time)
- Tomorrow we will start getting creative with these expression trees.
\[ X^2 + Y^2 = 1 \] (Here are the solved examples of expression trees)

\[ XY = 1 \]

\[ Y^2 = 1 + X^2 \]
\[ 2Y = XY^2 \]

\[ Y/2 = X/(1+X) \]

\[ Y/2 = (1 - X)/Z \]
\[ Y = X(90-Z) \]

\[ Y = B + C \]

\[ Y = A/B \]
\[ Y = \frac{1}{X^2} \]

\[ Y = C \left( \frac{A}{B} \right) \]

\[ Y = A \left( B + C \right) \]
**Pythagorean Identities:**

\[
\cos^2 \theta + \sin^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \\
1 + \cot^2 \theta = \csc^2 \theta
\]

**Reciprocal Identities:**

\[
\sin \theta = \frac{1}{\cos \theta} \quad \sin \theta \csc \theta = 1 \\
\csc \theta = \frac{1}{\sin \theta} \quad \cos \theta = \frac{1}{\sec \theta} \\
\cos \theta \sec \theta = 1 \quad \sec \theta = \frac{1}{\cos \theta} \\
\tan \theta = \frac{1}{\cot \theta} \quad \tan \theta \cot \theta = 1 \\
\cot \theta = \frac{1}{\tan \theta} \quad \tan \theta = \sin \theta / \cos \theta \\
\cot \theta = \cos \theta / \sin \theta
\]

**The cofunction Identities:**

\[
\sin \theta = \cos (90-\theta) \quad \cos \theta = \sin (90-\theta) \\
\tan \theta = \cot (90-\theta) \quad \cot \theta = \tan (90-\theta) \\
\sec \theta = \csc (90-\theta) \quad \csc \theta = \sec (90-\theta)
\]
Lesson Two: Semantic Tagging and Visualization

Objective
- Students will be able to add English meaning to the links of their tree diagrams.
- Students will be able to understand proximity, containment, and connectivity in the context of aesthetic computing.

Words to know: Proximity, Containment, Connectivity

Activities:
Objective One
- **Intro:** “Today we are going to learn how to turn these expression trees into works of art. I want to show you an example of what I will be asking you to do. I will give you a handout with more specifics at the end of class. This example uses the $y = mx + b$ equation. Each of you will be given a different equation for your project.”
- “Now let me explain a little about each of these examples. First, the dorm life example. This is an example of a ‘structure’. Each part of the equation is represented by something in the structure. Notice the expression tree in its basic form and then another one to the right of it with the words actually attached to the parts of the tree that mean something.” <Teacher can read through what each representation is using the key to the right of the second expression tree.>
- Teacher goes through the other two examples, explaining the ‘space’ (dorm) example and the ‘story’ (burger) example. Teacher should emphasize the words being placed on the expression tree, and the key that explains what each part of the equation is represented by.
- “So these are some examples of what will be expected of you for your project. Obviously, you may not have the resources to do this on the computer and that is okay. This project will be hand written for most of you. If you choose to do something like a story, you may not need pictures at all…the story would describe everything that is going on. You may use pictures if they pertain to your idea, but don’t box yourself in to one idea based on one particular picture you are dying to use.”
- “So let’s go through a few things before I give you the formal write-up of the project. If you notice in all these examples, there are categories of items in your equations. For example, in $y = mx+b$,
  - We have ‘m’ and ‘b’ as constants.
  - ‘x’ and ‘y’ are variables.
  - ‘+’ and ‘*’ are operations.
  - And ‘=’ is a relation.”
- Teacher then explicitly shows the students how each of the parts of the tree diagram have English semantics added to it. This is done by literally writing the
English words for these representations next to the corresponding parts on the tree diagram itself.

Objective Two

- "Now, there are three styles of metaphor that can help you in deciding what each part of your tree should be represented by. That is,
  - Containment - An example of containment would be something like addition being represented by two objects inside a container of some sort.
  - Proximity - An example of proximity is two objects next to each other representing multiplication. It’s something that is “next to,” “above,” “beside,” etc.
  - Connectivity - Connectivity two objects being connected by something to represent addition. The connection could be anything from a vine to a path or even just a line or string."
- “For your project, you will choose one of the trig identities to change. You will make it your structure, space, story, or whatever art form you choose to express this equation. The first step we did together in class.”
- “Please take those expression trees out for just a minute. (wait time for the students to get out their last assignments). Everyone should have his/her 'expression trees' out. When you change your equation from a regular equation to a trigonometric identity, there is just one adjustment you need to make to your expression tree. In the same way that a ‘+’ or ‘-‘ is treated as a function, the sin, cos, tan, sec, csc, and cot, needs to be treated as a function. For example, the expression tree for $\sin \theta = \frac{1}{\cos \theta}$, is as follows:

```
  =
 / \          
\ sin \                   
 /   \                  
\ θ \   \ cos \       
    /   
\ 1 \     
```
- “This should not be too big of an adjustment from the expression trees you have already completed, therefore, this should be the easiest part of your project.
- Teacher hands out the “Steps to get started” and the write up of the project.
- “The first page I gave you outlines some steps to get you started. The last step describes what you will be physically turning in to me at the end of this project. It will be different depending on what you decide to do for your project. You will then fill out the last page and turn it into me by tomorrow. Any questions?”
Aesthetic Computing: Teacher Resources

Dorm Life – Example of Structure

\[ y = mx + b \rightarrow y = m^*x + b \rightarrow (y, +(*(m, x), b)) \]
Aesthetic Computing: Teacher Resources

Dorm – Example of Space

\[ y = mx + b \rightarrow y = mx^2 + b \rightarrow (y, +(* (m, x), b)) \]
Joe is a chef at Burgeroo Fast Food Palace. He absolutely loves his job. Everyday he goes to work first thing in the morning and mixes the spices and meat together. Then after he had made all of the patties, he puts them in a frying pan and cooks them until they are well done. The next step is to prep the buns. Once Joe has cooked up all the burgers, he adds them to the buns and puts them in their special yellow wrapping. Everyone loves Joe’s burgers. In fact, you can always find a line of people outside Burgeroo Fast Food Palace who are just waiting for a bite!
Aesthetic Computing: Additional Examples

\[ \sin \theta = \cos (90 - \theta) \]

\[ \sin \theta = 1/\cos \theta \]

**Alien Invasion**

- is mother ship
- \(\sin\) is saucer
- \(\cos\) is saucer
- is escape shuttle

\(\theta\) is pilot

90 is alien invaders

branches represent inside

**60's Lamp**

- is light shade
- \(\sin\) is 60 watt socket
- \(\cos\) is 20 watt socket

/ is lamp base

\(\theta\) is 60/20 watt bulb

1 is switch

branches are wires inside lamp

note: “\(\sin\)” and “\(\cos\)” are assumed to be written on each saucer, differentiating them

note: the two lightbulbs are the same but each accepts different wattage
Steps to get started:

Step 1:
Create your tree diagram and make sure it is correct.

Step 2:
Write out a list of all the variables, constants, and operations.

Step 3:
Write out what you enjoy doing or some things you are interested in that you could use for this project. Use the following list of antonyms to help you or come up with some more of your own:

- shrink-grow
- leak out-pour in
- accumulation-decretion
- deepen-shallow
- weaken-strengthen
- extend-contract
- adjoin-separate
- expand upon-abbreviate
- spend-get paid
- give-receive
- expand-reduce
- development-destruction
- downgrade-upgrade
- accumulate-dispense
- connect-unfasten
- lose gas-refuel
- drink-fill up
- add color-black/white
- bloated-shriveled
- elevate-demote
- group-scatter
- balloon-shrivvel
- transparent-opaque
- clear-unclear

Step 4:
Decide what your theme will be (based on your interests in step 3) and how you will express that theme (through a story, a picture, a sculpture, etc).

Step 5:
Decide what will represent your variables, constants, and operations. Make sure that you choose the same category for similar types of operators. Think about the three types of metaphors: containment, proximity, or connectivity. Start making connections with these metaphors. Remember you can always use shapes and colors to represent these things as well.

Step 6:
Add English semantics to the tree diagram. In other words, label your tree diagram with the words you used to describe each part of the equation in step 5.

Step 7:
Write (or sketch) everything out. If it follows more of a story, describe the story. If it is represented more by a picture, sketch it out.

Step 8:
Type everything up, making sure to include:
1. Project title
2. Your name
3. Describe what each of the operations, constants, and variables are represented by.
4. Include a picture, sculpture, a 300 word description of your story, or any other form of art you managed to represent your equation with.
The equation I chose to do my project on is ____________________________________

The correct tree diagram for this equation is:

Some things I enjoy are:
______________________________________________________________________
______________________________________________________________________
______________________________________________________________________
______________________________________________________________________
______________________________________________________________________
______________________________________________________________________

A possible theme for my project is:
______________________________________________________________________
______________________________________________________________________
Aesthetic Computing: Teacher Resources

Lesson Four: Grading Rubric

Requirements:

- Students will turn in a written proposal for their project (6 pts)
  - Points based upon six questions given to the students to answer.
- Project is worked on during given class time (3 pts)
  - Three days maximum should be given in class, and points can be deducted for off-task behavior.
- Project is turned in on time (5 pts)
  - Points may be deducted for each day the project is late.
- Student uses good craftsmanship (5 pts)
  - Points can be deducted per major construction/cleanliness issue.
- All nodes are categorically consistent (8 pts)
  - Teacher may choose to give multiple points per correct node, or deduct for inconsistencies.
- Theme is apparent and consistent (5 pts)
  - Teacher may deduct per inconsistency.
- Student turns in both the mapping and design key (2 pts)
  - Teacher should deduct points for incomplete keys.
- Project conforms to proposal idea (6 pts)
  - Teacher may choose to deduct points per unnecessary change.

Total points possible = 40

Grading Scale

“A” = 40 to 36 points

“B” = 35 to 32 points

“C” = 31 to 28 points

“D” = 27 to 24 points

“F” = below 24 points
These tools allow the student and teacher the potential to interact with algebraic expressions. The 2D tool is specifically targeted toward moving numbers and variables for basic arithmetic, and illustrating the distributive property. The 3D tool may be used in one of two directions: given an algebraic expression, derive a 3D model, and vice versa.
2D Virtual Algebra Machine

This program is interactive and allows the student to view the distributive property of algebra. Parenthetic grouping is replaced by a circle, Numbers may be moved on top of one another to add or multiply. Dragging the “4” circle through the circle containing “2+3”:

![Image of 2D Virtual Algebra Machine](image)

distributes the “4”. The student may, for example, drag the 4 closer to the 2, which yields an “8”:

![Image of 2D Virtual Algebra Machine](image)
3D Virtual Algebra Machine

The 3D Virtual Algebra Machine is a piece of software that demonstrates some of the fundamental concepts behind Aesthetic Computing. The software takes traditional algebraic syntax and represents it with a 3D expression tree. The 3D tree is composed of nodes that are connected by tube-like edges. The resulting expression tree is a full binary tree (each node has zero or two children). Variables and constants (x, y, 4, -12) are children, and operators (+, -, *, /) are roots. Since all operators are roots that have two children, they are referred to as connectors. The set of connectors is defined in figure 1.

![Connectors]

A sample output can be seen in figure 2. This particular expression is 20(x + y).

![Expression Tree]
Figure 3 showcases the Distributive law of algebra, \( A(B + C) = AB + AC \). The software is still in its development phase so the variable names have been superimposed on the individual roots.

![Figure 3](image)

\[ A(B + C) = AB + AC \]

The engine behind the 3D virtual algebra machine will allow for custom renders of a given expression tree. This means high quality still images can be rendered with custom textures and camera angles. This is demonstrated in figure 4.

![Figure 4](image)

*Custom Renders*
To produce an expression tree the user will interface with a calculator-like control system. Figure 6 provides a digital sketch of the 3D Virtual Algebra Machine’s user-interface.