B-splines and the Riemann's Zeta Function on Integers

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1 Background and definitions

The Poisson's summation formula is:

$$\sum_{k\in\mathbb{Z}} f(kT) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}(n\frac{2\pi}{T}), \tag{1}$$

where $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)\exp\left(-i\omega x\right) dx.$$

The Poisson's summation formula holds for all integrable functions, given the series on the right hand side is absolutely convergent.

The simplest (centered) B-spline, β_0, is the characteristic function of $[-1/2, 1/2)$. Higher order B-splines β_m are defined recursively by $\beta_m = \beta_{m-1} * \beta_0$. Fourier transform of a B-spline is:

$$\hat{\beta}_m(\omega) = \operatorname{sinc}^{m+1}(\omega), \tag{2}$$

where $\operatorname{sinc}(t) := \sin(t/2)/(t/2)$.

Lemma 1 (Partition of Unity)**.**

$$\sum_{k\in\mathbb{Z}} \beta_m(k) = 1$$

Proof. Using Poisson's summation formula, setting $T = 1$, we have:

$$\sum_{k\in\mathbb{Z}} \beta_m(k) = \sum_{n\in\mathbb{Z}} \operatorname{sinc}^{m+1}(2\pi n) = 1,$$

since $\sin(\pi n) = 0$, for all $n \in \mathbb{Z}$ and the only non-zero term in the right hand side comes from $n = 0$: $\operatorname{sinc}(0) := 1$ that agrees with the actual limit of sinc at 0. □

1.1 Zeta function

The Riemann's Zeta function is defined as:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}. \tag{3}$$

2 Relationship between β_m and $\zeta(m+1)$

By playing around with the parameter T, one can recover parts of the series. The linear B-spline can be related to the $\zeta(2)$ and generally, β_m can be related to $\zeta(m+1)$. For the case of linear B-spline, by setting $T = 2$ in (1), we have:

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \beta_1(2k) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \operatorname{sinc}^2(n\pi)\\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \operatorname{sinc}^2(n\pi)\\ &= \frac{1}{2} + \frac{1}{(\pi/2)^2}\sum_{n=1}^{\infty} \frac{\sin^2(n\frac{\pi}{2})}{n^2}\\ &= \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin^2(n\frac{\pi}{2})}{n^2}.\end{aligned}$$

The last series is nearly $\zeta(2)$ except that $\sin^2(n\frac{\pi}{2})$ zeroes out the even terms. But it is, surely, convergent, since it is a sub-series of $\zeta(2)$, so as the series with remaining terms of $\zeta(2)$. The last series can be resolved, however, using $\zeta(2)$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin^2(n\frac{\pi}{2})}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}\\ &= \zeta(2) - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}\\ &= \zeta(2) - \frac{1}{4}\zeta(2)\\ &= \frac{3}{4}\zeta(2).\end{aligned}$$

Therefore, we can relate β_1 to $\zeta(2)$ by:

$$\sum_{k\in\mathbb{Z}} \beta_1(2k) = \frac{1}{2} + \frac{3}{\pi^2}\zeta(2). \tag{4}$$

Since the support of β_1 is $[-1, 1)$, the left hand side is 1 and we have:

$$\zeta(2) = \frac{\pi^2}{6}.$$

$\zeta(2)$ has been used in the so called **Basel problem**. Its inverse is the probability that two randomly selected integers are relatively prime.

2.1 Result

Using the above approach, one can derive the exact values for all even integers of ζ using evaluation of B-splines on even integers. Wikipedia lists $\zeta(2), \zeta(4)$ and $\zeta(6)$, but, apparently, evaluation of ζ, in general is difficult and there are papers for evaluation such as [1].

Is there a closed-form solution for all B-splines of odd order:

$$\sum_{k\in\mathbb{Z}} \beta_{2m+1}(2k) = ? \tag{5}$$

3 The difficult, but interesting case

The odd values of ζ, are interesting; for instance, $\zeta(3)$, known as **Apéry's constant** [2] is a curious number that occurs in various physical problems. The exact value of this constant is not known and it is an open problem whether this number is *transcendental*.

To derive Apéry's constant using the B-spline approach, we shall focus on the quadratic B-spline; employing the Poisson's sum (1), and $T = 2$, we have:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \beta_2(2k) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathrm{sinc}^3(\pi n) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin^3\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{2}\right)^3} \\
&= \frac{1}{2} + \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin^3\left(n\frac{\pi}{2}\right)}{n^3} \qquad \text{even terms are zero.}
\end{aligned}$$

The last series in the above derivation is more difficult to tackle since $\sin^3\left(n\frac{\pi}{2}\right)$ is alternating its sign for the non-zero terms.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sin^3\left(n\frac{\pi}{2}\right)}{n^3} &= \sum_{n=0}^{\infty} \frac{\sin^3\left((2n+1)\frac{\pi}{2}\right)}{(2n+1)^3} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \\
&= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3}
\end{aligned} \tag{6}$$

On the other hand:

$$\begin{aligned}\zeta(3) &= \sum_{n=1}^{\infty} \frac{1}{n^3} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{(2n)^3} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n^3} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \frac{1}{8}\zeta(3).\end{aligned}$$

Hence, we have:

$$\begin{aligned}\frac{7}{8}\zeta(3) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \\ &= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} + \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3}\end{aligned} \tag{7}$$

(7) and (6) are different and hence we can not resolve $\zeta(3)$ using this approach.

4 S.O.S

The question is: *is there a more appropriate choice than $T = 2$?* By choosing different values like $T = 3/2$, we get different sub-series of $\zeta(3)$. Can we build $\zeta(3)$, perhaps, from multiple choices for T?

References

[1] Jonathan M. Borwein, David M. Bradley, and Richard E. Crandall. Computational strategies for the Riemann zeta function. *J. Comput. Appl. Math.*, 121(1-2):247–296, 2000. Numerical analysis in the 20th century, Vol. I, Approximation theory.

[2] Wikipedia. Apéry's constant — wikipedia, the free encyclopedia, `http://en.wikipedia.org/wiki/Ap%C3%A9ry%27s_constant`, 2008. [Online; accessed 12-November-2008].