

COT4501, Solutions to Homework-6

03/01/09

2.5 Q17 Let $f = \frac{1}{t}$. We can use trapezoid rule to approximate logarithm function, since

$$I(f) = \ln(x) = \int_1^x \frac{1}{t} dt$$

1. Trapezoid rule on $I(f)$, with $a = 1$ and $b = x$, gives

$$\begin{aligned} T_n(f) &= \frac{(x-1)}{2n} (f(x_0) + 2f(x_1) \dots + 2f(x_{n-1}) + f(x_n)), \text{ where } x_k = 1 + (x-1)\frac{k}{n} \\ &= \frac{(x-1)}{2n} \left(1 + 2 \sum_{k=1}^{n-1} \frac{n}{n+k(x-1)} + \frac{1}{x} \right) = \frac{(x^2-1)}{2nx} + \sum_{k=1}^{n-1} \frac{x-1}{n+k(x-1)} \end{aligned}$$

2. Given $x \in [1, 2]$, the error bound

$$\begin{aligned} |I(f) - T_n(f)| &\leq \left(\frac{x-1}{12} \right) h^2 \max_{t \in [1, x]} |f^{(2)}(t)| \leq \frac{h^2}{12} \max_{t \in [1, 2]} \left| \frac{-2}{t^3} \right| \leq \frac{h^2}{6} \\ \frac{h^2}{6} &\leq \epsilon \Rightarrow h \leq \sqrt{6\epsilon} \Rightarrow n \geq \left\lceil \frac{1}{\sqrt{6\epsilon}} \right\rceil \end{aligned}$$

The terms required for an accuracy of $\epsilon = 10^{-8}, 10^{-15}$ are 4083 and 1.291×10^7 respectively.

3. See demo2517c.m; Yes, error is $3.749049293233497 \times 10^{-9} < 10^{-8}$

4. Given $x \in [1, \frac{3}{2}]$, the error bound

$$\begin{aligned} |I(f) - T_n(f)| &\leq \left(\frac{x-1}{12} \right) h^2 \max_{t \in [1, \frac{3}{2}]} \left| \frac{-2}{t^3} \right| \leq \frac{h^2}{24} 2 = \frac{h^2}{12} \\ \frac{h^2}{12} &\leq \epsilon \Rightarrow h \leq \sqrt{12\epsilon} \Rightarrow n \geq \left\lceil \frac{1}{\sqrt{12\epsilon}} \right\rceil \end{aligned}$$

The terms required for an accuracy of $\epsilon = 10^{-8}, 10^{-15}$ are 2887 and 9.12871×10^6 respectively.

5. We can use the method discussed in Problem-1.6.1. Any real number $x > 0$ can be uniquely represented as $x = \alpha \cdot \left(\frac{3}{2}\right)^\beta$, where $\alpha \in [1, \frac{3}{2}]$ and β is an integer.

$$\begin{aligned} \beta &= \left\lfloor \frac{\ln(r)}{\ln\left(\frac{3}{2}\right)} \right\rfloor, \quad \alpha = \frac{r}{\left(\frac{3}{2}\right)^\beta} \\ \ln(x) &= \ln(\alpha) + \beta \ln\left(\frac{3}{2}\right) \end{aligned}$$

Assume that $\ln(3/2)$ is known to arbitrary precision.

- 2.5 Q21.** Integration is the area under the curve. For a linear polynomial $f(x)$ from $x=a$ to $x=b$ the area under the curve is the area of a trapezium with height = $b-a$ and length of parallel sides = $f(a)$ and $f(b)$. So, the area is $(1/2)*(b-a)(f(a)+f(b))$, which is what is obtained from trapezoidal rule. Hence the proof.
- 2.5 Q22.** see proof of th. 2.3 on page 67 of textbook. In line 2 when we apply Th. 2.2 now what we will get is $-\sum_{i=1}^n \frac{1}{12} h_i^3 f''(\xi_{i,h_i})$. Note that h is now replaced by h_i . And so now, we just replace each h_i with the max over all h_i . Now we can take h_{max} outside the summation. And now the above expression will always be \leq the new expression that we obtain by replacing each h_i by the max over all h_i . Hence the proof follows.
- 2.5 Q24.** Almost exactly similar to what is proved on Page 70 of the textbook. The effect of rounding error is also stated underneath the proof.