COT4501, Solutions to Homework-4

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Some Taylor expansions:

$$f(x \pm h) = f(x) \pm h f^{(1)}(x) + \frac{h^2}{2!} f^{(2)}(x) \pm \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(x) + \mathcal{O}\left(h^5\right) \quad (1)$$

$$f(x \pm 2h) = f(x) \pm 2h f^{(1)}(x) + \frac{4h^2}{2!} f^{(2)}(x) \pm \frac{8h^3}{3!} f^{(3)}(x) + \frac{16h^4}{4!} f^{(4)}(x) + \mathcal{O}\left(h^5\right) \quad (2)$$

(2.2.3) see hw4_sol.m

(2.2.6) Answer to part (a) is already given. I will solve part (b).

$f(x-h) = f(x) - hf^1(x) + \frac{h^2}{2!} f^2(x) + \mathcal{O}(h^3)$ (3)

$f(x-2h) = f(x) - 2hf^1(x) + \frac{4h^2}{2!} f^2(x) + \mathcal{O}(h^3)$ (4)

4*(3) - (4):-

$4f(x-h) - f(x-2h) = 3f(x) - 2hf^1(x) + \mathcal{O}(h^3)$

Rearrange to get $f^1(x)$ in terms of others. So you get: $A = \frac{3}{2h}$, $B = -\frac{2}{h}$ and $C = -\frac{1}{h}$

(2.2.8) Upon substituting Taylor expansions for $f(x \pm h)$, $f(x \pm 2h)$, the error term becomes $\mathcal{O}(h^4)$.

(2.2.12) First in order to get a Taylor expansion of $f(x \pm h)$ at x, substitute $x \leftarrow x + h$ and $x_0 \leftarrow x$ in Eqn-1.1, to obtain:

$$f(x \pm h) = f(x) \pm h f^{(1)}(x) + \frac{h^2}{2!} f^{(2)}(x) \pm \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(x) + \mathcal{O}\left(h^5\right)$$

$$\Rightarrow f(x+h) + f(x-h) = 2f(x) + h^2 f^{(2)}(x) + \frac{2h^4}{4!} f^{(4)}(x) + \mathcal{O}\left(h^6\right)$$

$$\Rightarrow \left| \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - f^{(2)}(x) \right| \leq h^2 \left(\frac{2f^{(4)}(x)}{4!} + C_1 h^4 \right) \leq Ch^2$$

$$\Rightarrow f^{(2)}(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + \mathcal{O}\left(h^2\right)$$

We can approximate $f^{(2)}(x)$ using $f(x \pm h)$ and $f(x)$ since

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f^{(2)}(x)$$

Hence,

$$f^{(2)}(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + \mathcal{O}(h^2)$$

(2.2.22) Eqn-2.7 can be obtained by substituting Taylor expansions for $f(x \pm h)$. The derivative approximation to be constructed is given in 2.2.8 as:

$$f^{(1)}(x) = \frac{8\left(f(x+h) - f(x-h)\right) - f(x+2h) + f(x-2h)}{12h} + \mathcal{O}(h^4)$$