COT4501 Homework-3 Solutions

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1.6.1 The form $x = f \times 2^{\beta}$ is unique for some $\beta \in Z : (\beta - 1) < \log_2 x \leq \beta$ and $f \in [1/2, 1]$, where Z is the set of integers.

x	$\beta = \lceil \log_2 x \rceil$	$f = x/2^{\beta}$
13	4	.8125
25	5	.78125
$\frac{1}{3}$	-1	$.\overline{6}$
$\frac{1}{10}$	-3	.8

1.6.6 Let us examine if we can improve upon logarithm approximation using the form $z = \frac{1-x}{1+x}$

1. Let $p(x) = ln(1-x)$ and $q(x) = ln(1+x)$,written in terms of Taylor approximations and remainders as

$$\begin{aligned} p(x) &= p_n(x) + R_n^p(x) \\ q(x) &= q_n(x) + R_n^q(x), \; where \\ p_n(x) &= ln(1-x_0) - \sum_{k=1}^{n} \frac{1}{k} \left(\frac{x-x_0}{1-x_0} \right)^k \\ q_n(x) &= ln(1+x_0) + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \left(\frac{x-x_0}{1+x_0} \right)^k \end{aligned}$$

The remainders can be bound in a way similar to the method discussed in Section-1.6.

$$\begin{aligned} |R_n^p(x)| &= \left| \int_{x_0}^{x} (x-t)^n \, (1-t)^{-(n+1)} \, dt \right| \leq |x-x_0| \left| (1-t)^{-1} \right| \left| \frac{x-t}{1-t} \right|^n \\ |R_n^q(x)| &= \left| \int_{x_0}^{x} (x-t)^n \, (1+t)^{-(n+1)} \, dt \right| \leq |x-x_0| \left| (1+t)^{-1} \right| \left| \frac{x-t}{1+t} \right|^n \end{aligned}$$

For $g(t) = \left(\frac{x-t}{1-t}\right)$ and $h(t) = \left(\frac{x-t}{1+t}\right)$, as in Section-1.6, it can be seen that, there in no critical point for all $t \in (x_0, x)$. Hence, due to Extreme Value Theorem, the maximum clearly occurs at x_0 such that $|g(t)| \leq |g(x_0)|$ and $|h(t)| \leq |h(x_0)|$. We then have

$$\begin{aligned} |R_n^p(x)| &\leq |x - x_0| \left|(1-t)^{-1}\right| \left|\frac{x - x_0}{1 - x_0}\right|^n \\ |R_n^q(x)| &\leq |x - x_0| \left|(1+t)^{-1}\right| \left|\frac{x - x_0}{1 + x_0}\right|^n \end{aligned}$$

2. For $f(x) = ln\left(\frac{1-x}{1+x}\right)$, the Taylor expansion written as $f(x) = p_n^f(x) + R_n^f(x)$:

$$\begin{aligned} p_n^f(x) &= p_n(x) - q_n(x) \\ \left|R_n^f(x)\right| &= |R_n^p(x) - R_n^q(x)| \leq |R_n^p(x)| + |R_n^q(x)| \end{aligned}$$

3. Given $z \in [1/2, 1]$, $z = \left(\frac{1-x}{1+x}\right)$, x can be obtained from z as

$$x = g(z) = \left(\frac{1-z}{1+z}\right) \Rightarrow x \in [0, \frac{1}{3}] \Rightarrow x_0 = \frac{1}{6}$$

4. We can approximate $ln(z)$ using $f(x) = f(g(z))$. Substituting $x = \frac{1}{3}$, $x_0 = \frac{1}{6}$ and $t = \frac{1}{3}, 0$ in R_n^p, R_n^q respectively

$$\begin{aligned} |R_n^p(x)| &\leq |1/6| \left|\left(1 - \frac{1}{3}\right)^{-1}\right| \left|\frac{1/6}{5/6}\right|^n = \frac{5^{-n}}{4} \\ |R_n^q(x)| &\leq |1/6| \left|(1+0)^{-1}\right| \left|\frac{1/6}{7/6}\right|^n = \frac{7^{-n}}{6} \\ \left|R_n^f(x)\right| &\leq |R_n^p(x)| + |R_n^q(x)| \leq \frac{5^{-n}}{4} + \frac{7^{-n}}{6} \leq 10^{-16} \\ \implies n &\geq \mathbf{23} \end{aligned}$$

Note that the logarithm approximation in Section-1.6 required a degree-33 Taylor polynomial for the same accuracy. Since MATLAB stores only 16 significant digits, we cannot compute errors $\leq 10^{-16}$. So for an accuracy of 10^{-15}, we get **n≥21** from the above bound where as the numerical approach shows that **n≥19** is sufficient.

1.6.10 Given $f(x) = 1/x$, $x \in [1/2, 1]$, let us bound the Cauchy form of Remainder (Eqn-1.2), so as to determine the degree of Taylor polynomial necessary for an accuracy of 10^{-16}

$$
\begin{aligned}
R_n(x) &= \frac{1}{n!}\int\limits_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt = (-1)^{n+2}\int\limits_{x_0}^{x}(x-t)^n t^{-(n+2)}dt \\
\Rightarrow |R_n(x)| &= \left|\int\limits_{x_0}^{x}(\frac{x}{t}-1)^n t^{-2}dt\right| \leq |x-x_0|\left|t^{-2}\right|\left|\frac{x}{t}-1\right|^n, \text{ (as shown in Section} - 1.6)
\end{aligned}
$$

Since $g(t) = \frac{x-t}{t}$ does not have critical points with in $[x_0, x]$, due to Extreme Value Theorem, the maximum clearly occurs at one of the end points. Hence $|g(t)| \leq |g(x_0)|$. Substitute $x_0 = 3/4,\ x = 1,\ t = 1/2$, to obtain the upper bound as:

$$
\begin{aligned}
|R_n(x)| &\leq \left(1-\frac{3}{4}\right)\left(\frac{1}{2}\right)^{-2}\left(\frac{x}{x_0}-1\right)^n \leq \left(\frac{1}{3}\right)^n \leq 10^{-16} \Rightarrow n \geq \lceil 16/\log_{10}(3)\rceil = 34 \\
p_{34}(x) &= \sum_{k=0}^{34}\frac{(-1)^k(x-x_0)^k}{x_0^{k+1}}, \; where \, x_0 = \frac{3}{4}
\end{aligned}
$$