Numerical Analysis
COT4501
Spring 2009
Midterm I-Solutions

March 24, 2009

I Taylor series approximation [25 points]

1. [5 points] If you are given a function $f(x)$ and you seek its nth order Taylor series approximation $p_n(x)$ [and remainder $R_n(x)$] around x_0 and in the interval $[a, b]$, how do you go about doing this? Qualitatively, explain what conditions you think the function $f(x)$ should satisfy and the different forms of the remainder at your disposal. Do you think there is a relationship between the two forms of remainder?

   [2 points] for the first part if you give the formulation of Taylor series and the remainder.

   [1 point] for the conditions. It is important to state that f(x) as well as its n derivatives are continuou and differentiable over [a,b]. For most of you I took off 0.5 points if you didnt talk about the differentiabilty of the derivatives.

   [1 point] different forms of remainder: Cauchy form and the Lagrange form. Also give the mathematical form. Just stating the names is not sufficient.

   [1 point] The relationship is that both are equivalent and one can be derived from the other using the Integral Mean Value theorem.

2. [5 points] Work out the second-order Taylor-series approximation $p_2(x)$ for the function $f(x) = \exp\left\{-x^2\right\}$ at $x_0 = 0$ and in the interval $[a, b]$. What is the remainder $R_2(x)$? Write down the condition that $R_2(x)$ must satisfy for $|f(x) - p_2(x)| \leq \epsilon$, $\forall x \in [a, b]$.

   [2 points] The taylor series expansion is: $p_n(x) = 1 - x^2$

   [2 points] The remainder is: $R_2(x) = \frac{4}{3!}x^3\xi_x(2 - \xi_x^2)e^{-\xi_x^2} where \xi_x \epsilon (0, x)$

   [1 point] $|R_2(x)| < \epsilon$. For the condition to be satisfied $\forall x \epsilon [a, b]$,state that we need to find $max(R_2(x))$.

   Some people also solved it by using the series of $e^x$and then replacing $x$in the expansion with $-x^2$which is perfectly okay. If I took off your points for doing this bring it back to me and you "might" get your points back.

3. [5 points] What is the first order Taylor series approximation $p_1(x)$ for the function $f(x) = x^2 \log(x)$ [where log is the natural logarithm] in the interval $[0, 1]$ and for $x_0 = 0$? [Hint: You'll have to take the limits $\lim_{x\to 0} x^2 \log x$ and $\lim_{x\to 0} x \log x$ and use the $\frac{\infty}{\infty}$ form.] What is the remainder $R_1(x)$ and what is its worst (largest absolute) value in the interval $[0, 1]$? What is the error $E(x) = f(x) - p_1(x)$? Is there a discrepancy? Discuss.

   [2 points] $p_1(x) = 0$. Note that here $log(x) \to -\infty$as $x \to 0$.So you were required to tale the limit of the function at x=0 using the L'Hopital's rule.

   [1 point] $R_1(x) = \frac{1}{2}x^2(3 + 2ln(\xi_x))$

   [1 point] $|R_1(x)|$ is maximum when $\xi_x \to 0$ then $|R_1(x)| \to \infty$

   [1 point] There is no discrepancy because the error $E(x) = f(x) - p_1(x) = f(x) - 0 = f(x)$.and hence that the max abs remainder is infinite is perfectly okay.

4. [10 points] Given a function $f(x) = x^m \log x$ where m is an arbitrary integer greater than zero, find the order n for which the Taylor series approximation $p_n(x)$ at $x_0 = 0$ is non-zero. Is there a fundamental problem with finding such a Taylor series approximation? Discuss.

   [8 points] Keep differentiating and you will find that each derivative term of f(x) contains $x^{somepower}$term and $x^{somepower}ln(x)$ term. So for the second term again we have to take the limit at x = 0. Otherwise there is this fundamental difficulty that f(x) becomes infinity at x = 0 [2 points].

   Now take first derivative of x. We see that $f^1(x) = x^{m-1}(1 + \ln x)$. In this the first term that is x^{m-1}will become non zero at x = 0 if we differentiate it m-1 times, i.e., we have to have $f^m(x)$ which means n = m and hence the number of terms in the taylor expansion = m+1.

II **Derivative approximations [25 points overall]**

1. [5 points] What is the fundamental goal of the derivative approximation? Explain clearly the steps involved in beginning with a function $f(x)$ and ending up with a first derivative approximation for the function.

   Almost everyone got 5 points.

2. [5 points] If you are given the Taylor series approximation $p_n(x)$ for a function $f(x)$ at x_0, how does this change into an equivalent approximation for $f(x + h)$ in terms of $f(x)$? Write down the equivalences between (x, x_0) in $p_n(x)$ and $(x + h, x)$ in the approximation for $f(x + h)$.

   x is same as x+h and x_0 is same as x.

3. [5 points] Is the approximation $f'(x) \approx \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h}$ a valid first derivative approximation? [For a derivative approximation to be valid, $\lim_{h\to 0} \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h} - f'(x)$ should equal zero.]

   You have to evaluate the above limit and show that $\lim_{h\to 0} \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h} - f'(x) = 0$.This can be easily doen by expanding the two taylor series for $f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ and showing that the higher order terms tend to zero as x tends to zero.

4. [10 points] Begin with $f''(x) \approx Af(x + ah) + Af(x + bh) + Cf(x)$. Construct an approximation to $f''(x)$ by ensuring that i) the term involving $f(x)$ is zero, ii) the term involving $f'(x)$ is zero, and iii) that the coefficient of $f''(x)$ is one. Write down the resulting constraints involving A, C, a, b, and h. Pick a set of possible values for (A, C, a, b) that satisfy the constraints.

   $A[f(x + ah) + f(x + bh)] + Cf(x) = 2Af(x) + Cf(x) + Ah(a + b)f^1(x) + A\frac{h^2}{2}(a^2 + b^2)f^2(x) + ...$

   Applying the conditions we get

   $2A + C = 0$

   $A(a + b) = 0$

   $\frac{1}{2}A(a^2 + b^2) = 1$

   choose $A = 1$,$C = -2$,$a = 1$and $b = -1$. Now our A and C in the above equations are different than those in the question. So A = $1/h^2$ and C = $-2/h^2$

III **Linear Interpolation and Trapezoid Rule [25 points overall]**

1. [5 points] What is the relationship between the linear interpolation of a function $f(x)$ in the interval $[a, b]$ and the trapezoid rule. Conceptually relate the two.

   The trapezoid rule approximates the integral of the function by integrating over the linear interpolant of the function rather than the actual function itself.

2. [5 points] Construct a linear interpolation for $f(x) = \sin(x)$ for $x \in [0, \frac{\pi}{2}]$. What is the worst case error for this linear interpolation if you use the error bound?

   $y = \frac{(\frac{\pi}{2}-x)}{(\frac{\pi}{2}-0)}f(0) - \frac{(0-x)}{(\frac{\pi}{2}-0)}f(\frac{\pi}{2})$ or

   $y = \frac{(\frac{\pi}{2}-x)}{(\frac{\pi}{2}-0)}\sin 0 - \frac{(0-x)}{(\frac{\pi}{2}-0)}\sin\frac{\pi}{2} = \frac{2}{\pi}x$

   The worst case error is:

3. [5 points] Construct a trapezoidal rule integration of $I(f) = \int_0^{\frac{\pi}{2}} \sin(x)dx = -\cos x|_0^{\frac{\pi}{2}} = 1$.

   The trapezoid rule integration is the area under the linear interpolant:

   So, TI$_1$= $\frac{1}{2}(\frac{\pi}{2} - 0)(\sin\frac{\pi}{2} - \sin 0) = \frac{\pi}{4}$

4. [10 points] Given four points $A = (1, 5)$, B, C and $D = (2, 3)$ and the linear interpolation between B and C to be $3x + 4y = 5$ can you find the coordinates for B and C if you also know that the slope of the line joining A and B is 0.5 and that the slope of the line joining C and D is 2. (This kind of a problem arises in the spline literature.) Note that we know the equation of the line between B and C. [If you have difficulty solving simultaneous equations, just write down the conditions satisfied by the points B and C and you'll get a lot of partial credit.]

   Find the equation of AB:

   $y = 0.5x + c_1$This will satisfy point A(1,5). So,

   $5 = 0.5 * 1 + c_1$So, $c_1 = 4.5$.

   Hence, AB:$y = 0.5x + 4.5$

   Similarly find CD:

   $y = 2x + c_2$ This satisfies point D(2,3). So,

   $3 = 2 * 2 + c_2$ So, $c_2 = -1$

   Hence, CD:$y = 2x - 1$

   Now you get B by solving AB and BC. And you get C by solving BC and CD. the equation of BC is given to be : $3x + 4y = 5$

   It is uptill this point that you get 9/10 points. This is also what was mentioned in the question that if you have difficulty solving simultaneous equations then you can leave upto here and you get a lot of partial credit. If you also solve further then you get 10/10.

IV. Newton's method [25 points overall]

1. [5 points] Give a conceptual level explanation of Newton's method. Pick a function $f(x)$ and clearly explain what it is that the algorithm is trying to achieve.

   see the textbook.

2. [5 points] Apply Newton's method to find the root at the origin for the function $f(x) = \sqrt{x}$ with an initial condition $x_0 > 0$. Does the process converge?

   $x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = x_0 - 2x_0 = -x_0$. The process does not converge. Note that it is also not oscillatory because $-x_0$ because $x_0 > 0$ is negative and does not lie in the domain making $f(x)$ imaginary there. If you do not specify this or write it to be oscillatory instead you lose 2 points.

3. [5 points] Given a function $g(x)$ which is continuous and twice differentiable in the interval $(-\infty, \infty)$, use Newton's method to find the location α where the function attains its minimum. [You may assume if you wish that the function has only one minimum occurring at $x = 0$.]

   The Newton's method finds the roots of the equation. Now a function $g(x)$ attains minima when $g^1(x) = 0$. So, here we have to apply Newton's method on $g^1(x)$ and not $g(x)$.

   So, Newton's method will be:

   $x_{n+1} = x_n - \frac{g^1(x_n)}{g^2(x_n)}$

4. [10 points] Sketch a function that satisfies the following properties. i) The function $f(x)$ is continuous and differentiable in $(-\infty, \infty)$, ii) the function $f(x) = 0$ at $x = 0$, iii) $\frac{f(x)}{f'(x)} < 0$ for $x > \epsilon$ where $\epsilon > 0$ can be made arbitrarily small, iv) $\frac{f(x)}{f'(x)} > 0$ for $x < -\epsilon$. What happens when you run Newton's method on such a function? Qualitatively explain the different scenarios that unfold for different initial conditions x_0.

   See the scanned image that I will post tonight.