# Numerical Analysis
# COT4501
# Spring 2003
# Midterm II

30th April 2003

I [**25 points overall**]

[**10 points**] Find a polynomial of order 2 that interpolates at the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$, the following functions

1. [**5 points**] $f(x) = e^x$.

2. [**5 points**] $f(x) = x^2$.

You do not need to evaluate the exponential at any of the points. No calculators are required.

[**15 points**] Show that the Lagrange interpolation polynomial of order 2 for the function $f(x) = x^3$ at the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$ is $p_2(x) = x$.

1. [**2 points**] Why do you get a polynomial of degree 1 ($p_2(x) = x$) rather than a polynomial of degree 2?

2. [**10 points**] Assume an arbitrary polynomial approximation $q_2(x) = ax^2 + bx + c$. What are the conditions that $q_2(x)$ has to satisfy to be an approximating polynomial for $f(x) = x^3$at the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$? Can you obtain a solution that satisfies these conditions other than $q_2(x) = x$? Why?

3. [**3 points**] What is the $L_2$ norm of the error of the approximation $p_2(x) = x$ for $f(x) = x^3$?

II [**25 points overall**]

We are given a set of *nodes* $\{x_i,\ 0 \le i \le n\}$ at which to interpolate a given function $f(x)$ using a piecewise polynomial of maximum degree $d$. The problem is to find a piecewise polynomial $q_d(x)$ with degree of smoothness $N$. The degree of smoothness $N$ (not to be confused with the maximum degree $d$ of the polynomial) is defined as the number of derivatives of the function $q_d(x)$ that must be continuous at the *knot points* $\{x_k,\ 1 \le k \le (n-1)\}$.

1. [**5 points**] The number of degrees of freedom of the spline refers to the total number of free parameters that have to be specified in order to fully describe the piecewise polynomial $q_d(x)$.

   (a) Assuming $n$ subintervals, how many polynomials need to be specified?

   (b) How many free parameters need to be specified for each polynomial in each subinterval?

   (c) What is the total number of free parameters?

2. [**5 points**] The number of constraints can be broken down into i) extrinsic and ii) intrinsic factors.

   (a) *Extrinsic* constraints: How many interpolation conditions are present? Don't confuse this with the number of derivative smoothness conditions which is an intrinsic factor.

   (b) *Intrinsic* constraints: How many derivative smoothness constraints are present? Note that at this point we have not enforced extra derivative or other conditions at the two endpoints or elsewhere.

   (c) What is the total number of constraints?

3. [**5 points**] Show that for a cubic spline with $N = 2$, there are two extra degrees of freedom which have to be accounted for. Describe the two most common ways that the two extra constraints are imposed.

4. [**10 points**] Write down a general formula describing the difference between the number of free parameters and the number of constraints. Describe a way of compensating for the difference between the number of free parameters and the number of constraints for $d = 4$ and $N = 2$. The additional constraints imposed should not privilege any particular *knot* point. Are the constraints added by you intrinsic, extrinsic or both? Discuss.

III [**25 points overall**]

1. [**5 points**] Write down the approximating Lagrange interpolation polynomial of degree 2 ($p_2(x)$) which approximates a function $f(x)$ with exact equivalence at $x = a$, $x = b$, and $x = c$.

2. [**10 points**] Derive a general version of Simpson's rule by using $\int_a^b p_2(x)dx$ as an approximation for $\int_a^b f(x)\mathrm{dx}$ . Please note that $c \neq \frac{(a+b)}{2}$ at this point.

3. [**10 points**] For $f(x) = x^3$, does the general version of Simpson's rule give the correct result for $c \neq \frac{(a+b)}{2}$. Now specialize to the case $c = \frac{(a+b)}{2}$ to obtain the familiar Simpson's rule. Do you get the correct result when you substitute $c = \frac{(a+b)}{2}$ in your general Simpson's rule for $f(x) = x^3$? Why?

IV [**25 points overall**] Apply the following rules to obtain a numerical approximation to the integral $\int_0^2 e^x dx$. For each rule, write an expression for the absolute error. Assume just one interval $[0, 2]$. No calculators are necessary.

1. [**5 points**] Trapezoid rule.

2. [**5 points**] Simpson's rule.

3. [**5 points**] The midpoint rule.

4. [**10 points**] Extend the midpoint rule to include a second order Taylor series approximation around $x = c$ where $c = \frac{(0+2)}{2} = 1$. This extension to the midpoint rule involves integrating the second order Taylor series approximation of the function $f(x) = e^x$ term by term in the interval $[0, 2]$.

## List of Useful Formulae

**Taylor series approximation:** $f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^i f^{(i)}(x_0)}{i!} + \frac{(x-x_0)^{(n+1)} f^{(n+1)}(\xi_{[x_0,x]})}{(n+1)!}$ with $\xi_{[x_0,x]}$ in the interval $[x_0, x]$.

**Lagrange interpolation formula:** $p_n(x) = \sum_{k=0}^{n} f(x_k) L_k^{(n)}(x)$ where

$$L_k^{(n)}(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}.$$

$L_2$ **norm definition:** $\sqrt{\int_a^b (f(x) - q(x))^2 dx}$.

**Trapezoid rule:** $T(f) = \frac{1}{2}(b-a)(f(b) - f(a))$.

**Simpson's rule:** $S_2(f) = \frac{(b-a)}{6}(f(a) + 4f(c) + f(b))$ where $h = \frac{(b-a)}{2}$.

**Midpoint rule:** $M(f) = f(c)(b-a)$.