Practice problems on inverse trigonometric functions

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1. Simplify $\cos(2\tan^{-1} x)$.

   Let $y = 2\tan^{-1} x$. Then $x = \tan(y/2)$. Now LHS $= \cos(2\tan^{-1} x) = \cos 2y = 2\cos^2 y - 1$.
   Now $\cos y = 2\cos^2(y/2) - 1$. (Just as we got $\cos 2y = 2\cos^2 y - 1$).
   Now consider again $x = \tan(y/2)$, and hence $1 + x^2 = \sec^2(y/2)$, giving us $\cos^2(y/2) = 1/(1+x^2)$. Substituting this into the previous expression,
   we get LHS $= 2(2\cos^2(y/2) - 1)^2 - 1 = 2(\frac{2}{x^2+1} - 1)^2 - 1$.

2. Simplify $\tan(\sin^{-1} x)$.

   Let $y = \sin^{-1} x$, then $x = \sin y$. Now we have, LHS $= \tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1-\sin^2 y}} = \frac{x}{\sqrt{1-x^2}}$.

3. Find the domain and range of $g(x) = \sin^{-1}(3x + 1)$.

   Let $y = \sin^{-1}(3x + 1)$, then $\sin y = 3x + 1$. The range of $\sin y$ is $[-1, 1]$. Hence we have $-1 \leq 3x + 1 \leq 1$ and hence $-2 \leq 3x \leq 0$ and hence $-2/3 \leq x \leq 0$. Thus, the domain of $g(x)$ is $[-2/3, 0]$. The range is $[-\pi/2, \pi/2]$, because the domain of the sin function is restricted to be $[-\pi/2, \pi/2]$, so that it is a one-one function (and therefore has an inverse).