

Homework 4

July 30, 2008

The homework is due in class (BEFORE the lecture begins) on **Wednesday, 30th July**. As mentioned in the course syllabus, there are no electronic submissions and no late homeworks will be accepted unless you have an illness spanning the full period from the time the homework was assigned until it was due (and I shall need to see a medical practitioner's certificate to that effect). Standard academic honesty rules apply. You can discuss problems but the solutions turned in should be entirely your own. Cases of plagiarism will be dealt with strictly. Most of the problems are from the book by Stewart, 6th Edition.

1 Compulsory Problems

1. Section 3.1: Problems 18, 26, 27, 30, 32.

Problem 18: If $y = x^{1/3}$, then $y' = (1/3)x^{-2/3}$. Problem 26: $g'(u) = \sqrt{2} + \frac{\sqrt{3}}{2}u^{-1/2}$. Problem 27: $H'(x) = 3(x+1/x)^2(1-1/x^2)$. Problem 30: $v' = 2(\sqrt{x} + \frac{1}{x^{1/3}})(\frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-4/3})$. Problem 32: $y' = e^{x+1}$.

2. Section 3.1: Problems 53, 54, 57.

Problem 53: $y' = 18x^2 + 5$. If the curve $y = f(x)$ is to have a tangent with slope 4, then we must have $18x^2 + 5 = 4$ for some value of x , i.e. we must have $18x^2 = -1$. But this is impossible.

Problem 57: Let the equation of the tangent to $y = x^2 - 5x + 4$ be $y = mx + c$ where m is the slope and c the x-intercept. Now $m = dy/dx = 2x - 5$. The slope of the normal will be $\frac{-1}{m} = \frac{1}{5-2x}$. The given line is $x - 3y = 5$, i.e. $y = x/3 - 5/3$ and its slope is $1/3$. We know that this line is parallel to the normal, and hence the slope of the normal must be $1/3$. Hence we have $1/(5 - 2x) = 1/3$, i.e. $x = 1$. Hence we have $y = 1^2 - 5 \times 1 + 4 = 0$. Hence the equation of the normal is $y - 0 = (x - 1)/3$, i.e. $y = x/3 - 1/3$.

3. Section 3.2: Prove the quotient rule from first principles. Here is how to start: Let $F(x) = f(x)/g(x)$, then $F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$. Now, we have $F'(x) = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{hg(x)g(x+h)}$. Use tricks similar to the one we employed for the proof of the product rule in class, and complete the derivation.

$$F'(x) = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{hg(x)g(x+h)}.$$

Hence, by adding and subtracting $f(x)g(x)$, we have:

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{hg(x)g(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{hg(x)g(x+h)} \\ &= \lim_{h \rightarrow 0} \left(\frac{(f(x+h) - f(x))g(x)}{hg(x)g(x+h)} - \frac{f(x)(g(x+h) - g(x))}{hg(x)g(x+h)} \right) \\ &= \frac{\lim_{h \rightarrow 0} g(x)(f(x+h) - f(x))/h}{\lim_{h \rightarrow 0} g(x)g(x+h)} - \frac{\lim_{h \rightarrow 0} f(x)(g(x+h) - g(x))/h}{\lim_{h \rightarrow 0} g(x)g(x+h)} \quad (\text{this is by the quotient rule, because we are given } g(x) \neq 0). \end{aligned}$$

$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ (because for the denominators, $\lim_{h \rightarrow 0} g(x)g(x+h) = g(x) \times g(x)$; for the first numerator note that the limit exists because $f(x)$ is given to be differentiable and $g(x)$ is continuous (as $g(x)$ is given to be differentiable); similarly for the second numerator, the limit exists because $g(x)$ is given to be differentiable and $f(x)$ is continuous because it is given to be differentiable.)

4. Section 3.2: Problem 34. [You may use the quotient rule or the logarithm trick].

Let the tangent to the curve $y = \frac{\sqrt{x}}{x+1}$ at $(4, 0.4)$ be $y - 0.4 = m(x - 4)$. The slope m is equal to the value of y' at $x = 4$. Now $y' = \frac{1-x}{2\sqrt{x}(x+1)^2}$ using the quotient rule. At $x = 4$, we have $y' = -0.03 = m$. This gives us the equation of the tangent. The equation of the normal will be $y - 0.4 = -(x - 4)/m$ as it is perpendicular to the tangent.

5. Section 3.3: Problem 31 (You must use the quotient rule here, for practice).

$$\begin{aligned} \text{Part (a): } f'(x) &= \frac{(\sec^2 x)(\sec x) - (\sec x \tan x)(\tan x - 1)}{\sec^2 x} = \frac{\sec^3 x - (\tan x - 1)\sec x \tan x}{\sec^2 x} \\ &= \sec x - \frac{(\tan x - 1)\tan x}{\sec x} = \sec x - (\tan x - 1)\sin x \\ &= \sec x - \sin^2 x \sec x + \sin x = \cos x + \sin x. \end{aligned}$$

Part (b): $f(x) = \frac{\tan x - 1}{\sec x} = \sin x - \cos x$ (this is because $\tan x / \sec x = \tan x \cos x = \sin x$). Now $f'(x) = \cos x + \sin x$. This is the same as the derivative obtained in Part (a).

6. Section 3.4: Problems 10, 20, 38, 40, 46. You may use the logarithm trick here if you feel the need for it.

Problems 10: $y' = 2/3(x^4 + 1)^{-1/3}(4x^3)$. Problem 20: $y' = (x^2 + 1)(1/3)(x^2 + 2)^{-2/3}(2x) + (x^2 + 2)^{1/3}(2x)$ using the product rule combined with the chain rule. Problem 38: $y' = e^{k \tan \sqrt{x}} \frac{d}{dx} (k \tan \sqrt{x}) = k e^{k \tan \sqrt{x}} \sec^2(\sqrt{x}) \frac{d}{dx} \sqrt{x}$
 $= k e^{k \tan \sqrt{x}} \sec^2(\sqrt{x}) \frac{1}{2\sqrt{x}}$.

Problem 40: $y = \sin \sin \sin x$, hence $y' = \cos(\sin(\sin x)) \frac{d}{dx} (\sin \sin x) = \cos(\sin(\sin x)) \cos(\sin x) \cos x$.
 Problem 46: $y' = 4[x + (x + \sin^2 x)^3][1 + 3(x + \sin^2 x)^2(1 + 2 \sin x \cos x)]$.

7. Section 3.6: Problem 30.

$y = \log(\log(\log x))$). Let $u = \log(\log x)$, then we have $y = \log u$, and hence $\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx}$ using the chain rule.

Now let $v = \log x$, hence $u = \log v$. Then we have $\frac{du}{dx} = \frac{1}{v} \frac{dv}{dx} = \frac{1}{vx} = \frac{1}{x \log x}$. Putting this back into the first step, we have $\frac{dy}{dx} = \frac{1}{x \log x \log(\log x)}$. Note that you can do this all in one step if you wish, but then you run the risk of losing possibly all points if you've made an error.

8. Find the derivative of $f(x) = \sqrt{x}^{\sqrt{x}}$. Use the logarithm trick discussed in class.

Let $y = \sqrt{x}^{\sqrt{x}}$. Then $\log y = \sqrt{x} \log \sqrt{x} = \frac{1}{2} \sqrt{x} \log x$. Taking derivatives of both sides w.r.t. x , we have:

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left(\frac{\sqrt{x}}{x} + \frac{\log x}{2\sqrt{x}} \right) = \frac{1}{2} \left(\frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}} \right).$$

In the above step, note the usage of the product rule using the fact that the derivative of $\log x$ is $1/x$ and that the derivative of $\sqrt{x} = \frac{1}{2\sqrt{x}}$.

$$\text{Hence } \frac{dy}{dx} = \frac{1}{2} \sqrt{x}^{\sqrt{x}} \left(\frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}} \right).$$