Homework 2

July 16, 2008

1 Compulsory Problems

1. Section (2.2), problem 6.

   (a) 4, (b) 4, (c) 4, (d) not defined (see the graph), (e) 1, (f) -1, (g) does not exist (as left-hand and right-hand limits are unequal), (h) 1, (i) 2, (j) not defined (see graph), (k) 3, (l) not defined as the graph of the function oscillates wildly in the portion of the domain from 4 till less than 5.

2. Section (2.2), problem 12 (Note: Pay attention to all equality, inequality signs).

   If you plot the graph, you will see that $\lim_{x \to -1^-} = 3$ and $\lim_{x \to -1^+} = -1$, and hence $\lim_{x \to -1}$ does not exist. Similarly, $\lim_{x \to 1}$ also does not exist, as $\lim_{x \to 1^-} = 1$ and $\lim_{x \to 1^+} = 0$. However the left-hand limit and right-hand limit are equal to each other for all values of x such that $-1 < x < +1$, and for these values of x the limit does exist. Therefore, if $-1 < a < 1$, then $\lim_{x \to a}$ exists. This is because, for these values, $f(x) = x$. Also, the limit exists for all values of x that are less than -1, i.e. in the interval $(-\infty, -1)$, and similarly for all values in $(1, \infty)$.

3. Section (2.2), problem 14 (Note: Pay attention to all equality, inequality signs).

4. Section (2.3), problems 12, 22, 23, 28, 52, 58.

   Problem 12: $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)}$
   $= \lim_{x \to 7} \frac{(x+1)}{(x-1)} = 3/5$. Note that the cancellation of $x + 4$ could be done because $x \to 4, x \neq 4$.

   Problem 22: $\lim_{h \to 0} \frac{\sqrt{h+1}-1}{h} = \lim_{h \to 0} \frac{h}{h(\sqrt{h+1}+1)} = 0.5$. The second-last step is obtained by rationalizing the denominator. Note that the h cancels out from the numerator and denominator only because $h \to 0, h \neq 0$.

   Problem 23: Can be done very similarly, by rationalizing the numerator, i.e. multiplying both the numerator and denominator by $\sqrt{x+2} + 3$.

Problem 28: $\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h\to 0} \frac{1/(3+h)-1/3}{h} = \lim_{h\to 0} \frac{-h}{h(3+h)(3)}$
$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -1/9$. Again note the reason why h could cancel out.

Problem 52: $\lim_{v\to c^-} L_0\sqrt{1-v^2/c^2} = L_0 \lim_{v\to c^-} \sqrt{1-v^2/c^2} = L_0\sqrt{1-c^2/c^2} = 0$. The first step follows because L_0 is a constant independent of h, and the second step follows from the root-law of limits (i.e. law 11 on page 101). The physical interpretation of this result is that as the velocity of the object approaches that of light, the length of the object as perceived by the observer approaches 0. A left hand limit here is necessary because otherwise, L would cease to be a real number if $v > c$. Hence there is no right-hand limit.

Problem 58: Consider $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0} f(x)$ does not exist, and $\lim_{x\to 0} g(x)$ does not exist. But $\lim_{x\to 0} f(x) + g(x)$ does exist and is equal to 0. Here is one more example: $f(x) = 1/\tan(x)$ and $g(x) = -1/\sin(x)$. Then $\lim_{x\to 0} f(x)$ does not exist, and $\lim_{x\to 0} g(x)$ does not exist. But $\lim_{x\to 0} f(x) + g(x) = \lim_{x\to 0} \frac{1}{\tan x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$. This equals $\lim_{x\to 0} \frac{-2\sin^2(x/2)}{\sin x} = \lim_{x\to 0} \frac{\frac{-2\sin^2(x/2)}{x}}{\frac{\sin x}{x}}$. This further equals $[\lim_{x\to 0} -2\sin(x/2)][\lim_{x\to 0} \frac{\sin(x/2)}{x}]/[\lim_{x\to 0} \frac{\sin x}{x}]. = -2(0)(1)/(1) = 0$. Carefully note how the product and quotient laws were applied here. Note that they could be applied only because the individual limits existed.