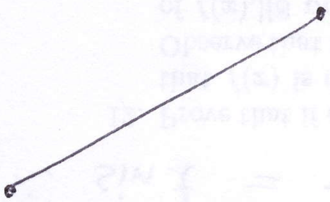
9. Suppose that a particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point $(2,3)$, suppose that its y-coordinate increases at the rate of 4 units per second. What is the rate of change of its x coordinate. **[7 points]**

$$\frac{dy}{dt} = \frac{1}{2\sqrt{1+x^3}} \, 3x^2 \frac{dx}{dt}$$

$$\cancel{4} = \frac{1}{2\sqrt{1+2^3}} \, 3(2)^2 \frac{dx}{dt}$$

$$\frac{1}{\cancel{3}} = \frac{1}{2(3)} \frac{dx}{dt} \quad \therefore \frac{dx}{dt} = 2 \text{ units/sec}$$

10. Consider a point $A \equiv (0,0)$ and $B \equiv (3,3)$. Suppose a person walks from A to B along line AB at a speed of 0.3 units per second, whereas another person walks from B to A along line BA at a speed of 0.4 units per second. How fast is the distance between these people changing after 1 second? Assume that both started off at the exact same time. **[7 points]**



After t seconds, the coordinates of person 1 are $P_1(t) = (0.15\sqrt{2}t, 0.15\sqrt{2}t)$

as he travels $0.3t$ units & so

$$\sqrt{x^2+y^2} = 0.3t \quad \therefore \sqrt{2x^2} = 0.3t$$

$$\therefore x = \frac{0.3\sqrt{2}t}{2} = 0.15\sqrt{2}t$$

Sim. for person 2, $\sqrt{(x-3)^2+(y-3)^2} = -0.4t$

$\therefore \sqrt{2}(x-3) = -0.4t$

$\therefore x-3 = -0.2\sqrt{2}t$

$\therefore x = 3-0.2\sqrt{2}t$

$\therefore P_2(t) = (3-0.2\sqrt{2}t, 3-0.2\sqrt{2}t)$

9. Suppose that a particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point $(2,3)$, suppose that its y-coordinate increases at the rate of 4 units per second. What is the rate of change of its x coordinate. **[7 points]**

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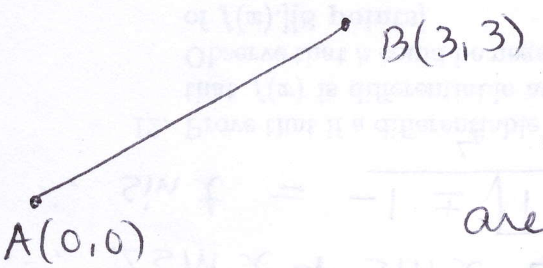

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$$\therefore \sqrt{2}\,(x-3) = -0.4t$$

$$\therefore x-3 = -0.2\sqrt{2}t$$

$$\therefore x = 3-0.2\sqrt{2}t$$

$$\therefore P_2(t) = (3-0.2\sqrt{2}t, \; 3-0.2\sqrt{2}t)$$

13. The following two questions will force you to think creatively. (1) Suppose $f(x)$ is a polynomial of degree n. How many times can its graph intersect the x-axis at the most? How many critical points can it have at the most? A critical point of a function is a point at which its first derivative is zero or does not exist. Now consider that $g(x)$ is another polynomial with degree m where $m > n$. Find $\lim_{x\to\infty} f(x)/g(x)$. What is the value of this limit if $m = n$? (2) A function $f(x)$ is said to have a fixed point if there is a number c in its domain such that $f(c) = c$. Use the intermediate value theorem to prove that any continuous function with domain $[0, 1]$ and range which is a sub-interval of $[0, 1]$ (like say $[0.2, 0.3]$) will always have a fixed point.**[5 + 5 = 10 points]**.

① $f(x)$ can have atmost n roots, so it will intersect the X axis at most n times. $f'(x)$ has degree $n-1$, so $f'(x)$ has at most $n-1$ roots. So $f(x)$ has $n-1$ critical points ($f(x)$ is defined everywhere on the real line as it is a polynomial on an unbounded domain).

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\left(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_n x^n + \cdots b_1 x + b_0\right)}$$

$= 0$

If $m = n$, this limit will be some non-zero constant $\frac{a_n}{b_m}$. If $m < n$, this limit is ∞.

② Consider the function $g(x) = f(x) - x$. This is continuous over $[0,1]$. So by intermediate value theorem, there is some x such that $g(x) = 0$ & hence $f(x) = x$.

Also $g(0) < 0$ and $g(1) > 0$.

$$d(P_1(t), P_2(t)) = \sqrt{\left(0.15\sqrt{2}t - (3 - 0.2\sqrt{2}t)\right)^2 + \left(0.15\sqrt{2}t - (3 - 0.2\sqrt{2}t)\right)^2}$$

Problem 10 contd

$$= \sqrt{2}\left(0.35\sqrt{2}t - 3\right)$$

$$\frac{d\, d(P_1, P_2)}{dt} = 0.7 \longrightarrow \text{indep of time}$$

$\therefore$ dist. betn. P_1 & P_2 changes by 0.7 sq. units after 1 second.

Problem 11 contd.

If $\sin t = -1$, $\therefore t = -\pi/2$. If $\sin t = 1/2$, $t = \pi/6$

$\therefore t = \pi/6$ as $-\pi/2$ lies outside $[0, \pi/2]$

$$f(\pi/6) = 2\cos \pi/6 + \sin \pi/3$$

$$= 2(\sqrt{3}/2) + \sqrt{3}/2 = 3\sqrt{3}/2$$

$$= 3 \times 0.866 = 2.598$$

abs. max is $x = \pi/6$

abs. min is $x = \pi/2$