Homework 6
(due Friday, April 21, 2006)

April 14, 2006

1. **Mean field theory:** Consider the objective function

$$E(s) = -\left[\frac{1}{2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} J_{mn} s_m s_n + \sum_{n=0}^{N-1} H_n s_n\right] \tag{1}$$

with $s_m \in \{-1, 1\}, \forall m$. In (1). (In the above, $m \oplus 1 = 0$ when $m = N-1$ and $m \ominus 1 = N-1$ when $m = 0$.) $J_{mn} = 0$ when $n \neq m \oplus 1$ or $n \neq m \ominus 1$. The corresponding probability distribution is

$$\Pr(S = s) = \frac{\exp\{-\beta E(s)\}}{Z(\beta)}. \tag{2}$$

- If the free energy $F \stackrel{\text{def}}{=} -T \log Z$ where $T = \frac{1}{\beta}$, evaluate $\frac{\partial F}{\partial T}$ and show its relationship to the entropy. What does this relationship mean conceptually?
- Relate $\frac{\partial \log Z}{\partial h_i}$ and $\frac{\partial^2 \log Z}{\partial h_i^2}$ to the mean and variance of s_m. Construct corresponding derivatives to express the mean and variance of the product $s_m s_n$. What do these relationships mean conceptually?
- Assume a mean field approximation wherein we construct a second distribution $Q(s) = \frac{\exp\{-\beta \sum_{m=0}^{N-1} r_m s_m\}}{\prod_{m=0}^{N-1}(1+\exp\{-\beta r_m\})}$ and determine the best $\{r_m\}$ that minimizes the Kullback-Leibler (KL) divergence $D(Q||P) = \sum_{\{s\}} Q(s) \log \frac{Q(s)}{P(s)}$. What is the optimized value of $D(Q||P)$ in terms of the optimal value of $\{r_m\}$?