Homework 4
(due Tuesday, March 7th, 2006)

March 2, 2006

1. Imagine that you are in charge of a political survey where each subject is classified according to four category spaces—a) conventional (C): {liberal (0), conservative (1)}, b) nonlinear (N): {moderate (0), radical (1)}, c) orthogonal (O): {authoritarian (0), libertarian (1)} and d) moral (M): {exclusivist (0), inclusivist (1)}. In each space, please note that we have assigned binary values to the labels. For example, liberal is assigned '0' and conservative '1' etc. We are interested in studying the co-occurrences between these spaces.

- Given a pool of candidates who have been classified according to the four category structures above, how would you estimate the joint probability $\Pr(C, N, O, M)$ between the four spaces from the data? [Explain qualitatively how you'd build up the four-way probability distribution.]

- You are given the following: $\Pr(C = 0) = x$, $\Pr(N = 0) = y$, $\Pr(O = 0) = z$, and $\Pr(M = 0) = w$. Also, $\Pr(C = 0, N = 0) = a$, $\Pr(N = 0, O = 0) = b$, $\Pr(O = 0, M = 0) = c$, and $\Pr(M = 0, C = 0) = d$. Evaluate the pairwise joint probabilities $\Pr(C, N)$, $\Pr(N, O)$, $\Pr(O, M)$ and $\Pr(M, C)$ given this information. [You'll need to use basic rules relating two variable probability distributions to single variable probability distributions. Pretend that a, b, c, d and x, y, z, w are numbers. Now, write all the probabilities in terms of these 8 numbers. You'll need to know that $\sum_c \Pr(C = c, N) = \Pr(N)$.]

- Given the above pairwise probabilities, estimate the full joint probability $\Pr(C, N, O, M)$ two ways. In case 1, remove $\Pr(M, C)$ to get a tree. In case 2, remove $\Pr(C, N)$ to get a tree. List the conditional probability approximations in both cases. Write down all 16 possibilities for both cases. [Warning: This will take some time. However, since the Fall 2002 class completely botched this question, I'm making sure that if I asked you ten years from now to answer this question, you'll do it like a zombie. We're going for permanent memory etching here. If you think this is horse%$#@, please realize that it builds character.]

- For both cases above, evaluate $\Pr(C, O)$ which was not given to you. [Establish expressions such that both approaches give you the same answer for all four possibilities. You cannot assume that you have the co-occurrences from the data from which to estimate $\Pr(C, O)$.]

2. Show the equivalence of the following free energy to the Bethe free energy

$$\begin{aligned}F_{\text{equiv}}(\{p_{ij}, p_i, \sigma_{ij}, \rho_i, \gamma_{ij}, \lambda_{ij}\}) &= \sum_{ij:i>j} \sum_{x_i, x_j} p_{ij}(x_i, x_j) \log \frac{p_{ij}(x_i, x_j)}{\sigma_{ji}^{\delta}(x_i) \sigma_{ij}^{\delta}(x_j)} \\ &+ \delta \sum_{ij:i \neq j} \sum_{x_i} \sigma_{ij}(x_j) + \sum_i \sum_{x_i} p_i(x_i) \log \frac{p_i(x_i)}{\rho_i^{q_i}(x_i)} + \sum_i q_i \sum_{x_i} \rho_i(x_i) \\ - \sum_{ij:i>j} \sum_{x_i, x_j} & p_{ij}(x_i, x_j) \log \psi_{ij}(x_i, x_j) \psi_i^{\xi_i}(x_i) \psi_j^{\xi_j}(x_j) - \sum_i r_i \sum_{x_i} p_i(x_i) \log \psi_i(x_i) \\ + \sum_{ij:i>j} \sum_{x_j} \lambda_{ij}(x_j) [\sum_{x_i} p_{ij}(x_i, x_j) - p_j(x_j)] &+ \sum_{ij:i>j} \sum_{x_i} \lambda_{ji}(x_i) [\sum_{x_j} p_{ij}(x_i, x_j) - p_i(x_i)] \\ &+ \sum_{ij:i>j} \gamma_{ij} (\sum_{x_i, x_j} p_{ij}(x_i, x_j) - 1). \quad (1)\end{aligned}$$

What are the fixed point equations for $\{p_{ij}, p_i, \sigma_{ij}, \rho_i\}$? [You'll have to provide the constraints on δ, q_i, r_i, ξ_i in order to establish equivalence.]