Solutions to Homework 1

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2.27. We can see that

$$\begin{aligned}
& H[(p_1+p_2+\cdots+p_m),(p_{m+1}+p_{m+2}+\cdots+p_I)] \\
& +(p_1+\cdots+p_m)H\left(\frac{p_1}{(p_1+\cdots+p_m)},\cdots,\frac{p_m}{(p_1+\cdots+p_m)}\right) \\
& +(p_{m+1}+\cdots+p_I)H\left(\frac{p_{m+1}}{(p_{m+1}+\cdots+p_I)},\cdots,\frac{p_I}{(p_{m+1}+\cdots+p_I)}\right) &=\\
& -\frac{\sum_{i=1}^m p_i}{\sum_{i=1}^I p_i}\log\frac{\sum_{i=1}^m p_i}{\sum_{i=1}^I p_i}-\frac{\sum_{i=m+1}^I p_i}{\sum_{i=1}^I p_i}\log\frac{\sum_{i=m+1}^I p_i}{\sum_{i=1}^I p_i} \\
& -\sum_{i=1}^m p_i\left[\sum_{k=1}^m\frac{p_k}{\sum_{j=1}^m p_j}\log\frac{p_k}{\sum_{j=1}^m p_j}\right]-\sum_{l=m+1}^I p_l\left[\sum_{n=m+1}^I\frac{p_n}{\sum_{o=m+1}^I p_o}\log\frac{p_n}{\sum_{o=m+1}^I p_o}\right] &=\\
& -\sum_{i=1}^m p_i\log\sum_{i=1}^m p_i-\sum_{i=m+1}^I p_i\log\sum_{i=m+1}^I p_i-\sum_{k=1}^m p_k\log\frac{p_k}{\sum_{j=1}^m p_j}-\sum_{n=m+1}^I p_n\log\frac{p_n}{\sum_{o=m+1}^I p_o} &=\\
& -\sum_{k=1}^m p_k\log p_k-\sum_{n=m+1}^I p_n\log p_n &=\\
& -\sum_{i=1}^I p_i\log p_i &= H(\mathbf{p}).(1)
\end{aligned}$$

2.29. Probability of getting $(x-1)$ tails and one head is $(1-p)^{x-1}p$ where $\Pr(\text{Heads})=p$. The entropy S is

$$\begin{aligned}
-\sum_{x=1}^{\infty}p(1-p)^{x-1}\log[p(1-p)^{x-1}] &=\\
-p\log p\sum_{x=1}^{\infty}(1-p)^{x-1}-p\log(1-p)\sum_{x=1}^{\infty}(x-1)(1-p)^{x-1} &=\\
-\log p-p\log(1-p)\sum_{x=1}^{\infty}(x-1)(1-p)^{x-1}. \qquad (2)
\end{aligned}$$

The summation

$$\begin{aligned}
\sum_{x=1}^{k}(x-1)(1-p)^{x-1} &=\\
\sum_{x=1}^{k}x(1-p)^{x-1}-\frac{1-(1-p)^k}{p}.
\end{aligned}$$

Let the summation $R=\sum_{x=1}^k x(1-p)^{x-1}$. Then $(1-p)R=\sum_{x=1}^k x(1-p)^x=\sum_{x=2}^{k+1}(x-1)(1-p)^{x-1}$. Subtracting term by term, we get $pR=\sum_{x=0}^k(1-p)^x-k(1-p)^k=\frac{[1-(1-p)^{k+1}]}{p}-k(1-p)^k$. When we set

k to ∞, we get $R = \frac{1}{p^2}$. Inserting this in (2), we get

$$\begin{aligned} S &= -\log p - p \log(1-p) \left[\frac{1}{p^2} - \frac{1}{p}\right] \\ &= -\frac{1}{p}[p \log p + (1-p)\log(1-p)]. \end{aligned}$$

2.30. Let B_1 be the random variable corresponding to the first ball and B_2 the random variable corresponding to the second ball. Then the probability that the first ball drawn is white is $\Pr(B_1 = \text{white}) = \frac{w}{w+b}$. For the second ball,

$$\begin{aligned} \Pr(B_2 = \text{white}) &= \Pr(B_2 = \text{white}|B_1 = \text{white}) \Pr(B_1 = \text{white}) + \Pr(B_2 = \text{white}|B_1 = \text{black}) \Pr(B_1 = \text{black}) \\ &= \frac{w-1}{w+b-1} \frac{w}{w+b} + \frac{w}{w+b-1} \frac{b}{w+b} \\ &= \frac{w}{w+b} \\ &= \Pr(B_1 = \text{white}). \end{aligned}$$

2.32. The location and orientation of the line segment are independent random variables. If $a < b$, then the set of locations of the line segment are drawn from a uniform distribution in the interval $[0, b]$ and the set of orientations are drawn from a uniform distribution in $[0, \pi]$ (since the line segment is not oriented). If the lines are drawn parallel to the x-axis, the length of the line segment in the y-direction is $a\sin(\theta)$. The probability of the line segment crossing a line is

$$\int_0^{\pi} \int_0^{a\sin(\theta)} \frac{1}{\pi b} dt d\theta = \frac{2a}{\pi b}.$$

If $a \geq b$, we divide the situation into whether $a\sin(\theta) > b$ or $a \sin\theta \leq b$. The probability of the line segment projection crossing the line is 1 if $a\sin\theta > b$. If $a \sin\theta \leq b$, then the probability of crossing the line is $\frac{a\sin\theta}{\pi b}$ (for every fixed value of θ) from before. Therefore the total probability is

$$\begin{aligned} 2\int_0^{\arcsin(\frac{b}{a})} \int_0^{a\sin(\theta)} \frac{1}{\pi b} dt d\theta + 2\int_{\arcsin(\frac{b}{a})}^{\frac{\pi}{2}} \int_0^{b} \frac{1}{\pi b} dt d\theta \\ = \frac{2a - 2\sqrt{a^2 - b^2}}{\pi b} + \frac{\pi - 2\arcsin(\frac{b}{a})}{\pi} \\ = \frac{2\left[\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1} + \arccos(\frac{b}{a})\right]}{\pi} \end{aligned}$$