Lecture 5

Cont. var.

$$E(X) = \int_{-\infty}^{+\infty} X(x)\, p(x)\, dx$$

$\downarrow$

mapping $\Omega \to \mathbb{R}$

Usually

$X(x) = x$ — identity function

all the complexity is on $p(x)$

Ex: Normal distribution.

$X \overset{d}{\sim} N(0,1)$

Variance & Covariance

$$Var(X) = E\left[(X - E(X))^2 \right]$$

$$= E\left[X^2 - 2X E(X) + E^2(X) \right]$$

$$= E(X^2) - 2E(X)E(X) + E^2(X)$$

$$= E(X^2) - E^2(X)$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

Properties

1° $Var(X) = Cov(X,X)$

2° $Var(c) = 0$

3° $Var(-X) = a^2 Var(X)$

4° $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

$E[E[X]] = E[X]$

$Y = X$

$Var(2X) = 2^2 Var(X) + 2Cov(X,X)$

Covariance is maximum.

$Y = -X$

$Var(X+Y) = 0$

Y indep of X

$Cov(X,Y) = 0$

$Var(X+Y) = Var(X) - Var(Y)$

$\forall X_i$ independent

$Var(\sum X_i) = \sum Var(X_i)$

i.i.d variables

Covariance is b_i-linear
→ linear in both arguments

$$\mathrm{Cov}\left(\sum_i a_i X_i, \sum_j b_j Y_j\right) =$$

$$= \sum_i a_i \mathrm{Cov}\left(X_i, \sum_j b_j Y_j\right)$$

$$= \sum_i \sum_j a_i b_j \mathrm{Cov}(X_i, Y_j)$$

Var + independence
⇓
~~linear~~
semi-linear (+)
not scalar multiple

Conditional Expectation

$$E[X|A] = \frac{\sum_{\omega \in A} X(\omega) P(\{\omega\})}{P(A)}$$ - discrete r.v.

$A = \{X = x\}$ $\quad P(A) \neq 0$

$A \to$ Reals

Cont. r.v.

$$E[X|A] = \frac{\int_A X(x) p(x)\,dx}{P(A)}$$

$P(A) = \int_A p_X\,dx$

Random Vectors

$$\mathbb{X} = (X_1, X_2, \ldots, X_n)$$

$$E(\mathbb{X}) = [E(X_1), \ldots, E(X_n)]$$

$$V_n(\mathbb{X}) = \begin{pmatrix} V_n(X_1) & \cdots & C_n(X_1, X_2) \\ \vdots & \ddots & \vdots \\ & & V_n(X_n) \end{pmatrix}$$

$$= E(\mathbb{X}^T \mathbb{X}) - E(\mathbb{X})^T E(\mathbb{X})$$

$X^T X$

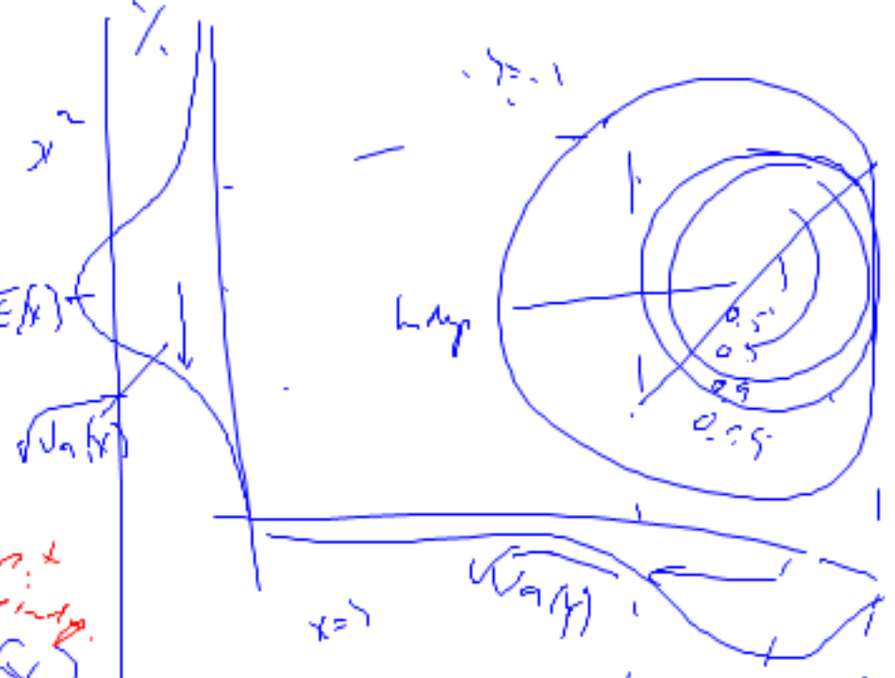
(X, Y)

(X', Y')

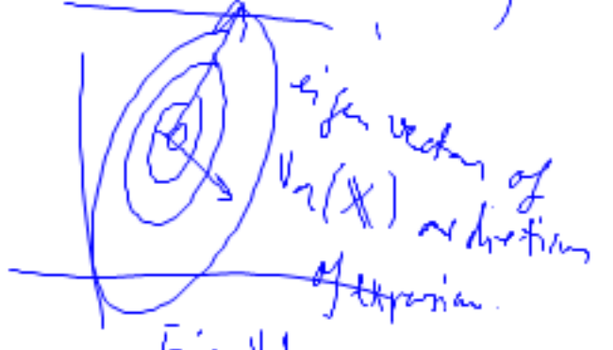
$X' = aX + bY$

$Y' = cX + dY$

Q: a, b, c, d such that X', Y' are uncorrelated.



General Case:



Eigen vectors of $V_n(\mathbb{X})$ are directions of ellipsoid.

Eigen values are ellipsoid axes.