Lecture 3

Random Variables

Mapping: $X: \Omega \to \mathbb{R}$

We want to determine things like

$P\{X < x\}$ for a given $x \in \mathbb{R}$ ← cont. dist.

$P\{X = x\}$ - discrete distribution

Q: What event is $X < x$

$X < x$ to be an event, it has to be the case that

$X < x$ is in $\mathcal{F}$

$$X < x \equiv \{\omega \in \Omega \mid X(\omega) < x\}$$

A mapping $X: \Omega \to \mathbb{R}$ is a R.V. iff

$\forall x \in \mathbb{R}$ $\{\omega \in \Omega \mid X(\omega) < x\} \in \mathcal{F}$

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathbb{Q} \\ 1 & \text{if } \omega \in \mathbb{I} \end{cases}$$

$x = \frac{1}{2}$

$X < \frac{1}{2} = \mathbb{Q}$

Thm: If X is continuous except in a countable # of points then X is an R.V.

Discrete Random Variables
Continuous

$X(\omega),\ \forall \omega \in \Omega$

$X(x),\ \forall x \in \mathbb{R}$

$X(x) = x$

$P[X=x]$

$P[X<x] = \int_{-\infty}^{x} p(t)\,dt$

$= \sum_{\substack{\omega\in\Omega \\ X(\omega)=x}} P(\{\omega\})$

$\downarrow$

p_ω

Expectation

Let X be a 0, 1 R.V. over a discrete space $\Omega = \{1, 2, \dots, n\}$

$P[X=0] = \sum_{\substack{\omega\in\Omega \\ X(\omega)=0}} p_\omega$

$P[X=1] = \dots$ $X(\omega)=1$

Intuitively:

$E[X] = 0 \cdot \cancel{P[X=0]} + 1 \cdot P[X=1]$

$= P[X=1]$

$\longrightarrow$ in general for discrete r.v.

$E(X) = \sum_{x \in \{X(\omega) | \omega \in \Omega\}} x \cdot P[X=x]$

Continuous R.V.

$E[X] = \int_{-\infty}^{+\infty} x \cdot p(x)\,dx$

Properties of expectation

1° $\forall c \in \mathbb{R}$ (constant)

$E[c] = c$

2° Linearity

$\forall a \in \mathbb{R}\quad E[aX] = a \cdot E[X]$

$\forall X, Y$ r.v.

$E(X+Y) = E(X) + E(Y)$

$E\left[\sum_i X_i\right] = \sum_i E(X_i)$

Do not require independence.

Independence of Random Variables

X, Y are independent iff $\forall x, y \in \mathbb{R}$

$X < x$ is independent from $Y < y$

$\{\omega \mid X(\omega) < x\}$ — " — $\{\omega \mid Y(\omega) < y\}$

Compare $\Downarrow$ iff

$E[XY] = E[X] \cdot E[Y]$

a_i b_j