Topological Relationships between Complex Spatial Objects

Markus Schneider
University of Florida
Department of Computer &
Information Science & Engineering
Gainesville, FL 32611, USA
mschneid@cise.ufl.edu

Thomas Behr
FernUniversität Hagen,
Praktische Informatik IV
D-58084 Hagen, Germany
thomas.behr@fernuni-hagen.de

Abstract

For a long time topological relationships between spatial objects have been a focus of research in a number of disciplines like artificial intelligence, cognitive science, linguistics, robotics, and spatial reasoning. Especially as predicates they support the design of suitable query languages for spatial data retrieval and analysis in spatial databases and geographical information systems (GIS). Unfortunately, they have so far only been defined for and applicable to simplified abstractions of spatial objects like single points, continuous lines, and simple regions. With the introduction of *complex spatial data types* an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. This paper closes this gap and first introduces definitions of general and versatile spatial data types for *complex points*, *complex lines*, and *complex regions*. Based on the well known 9-intersection model, it then determines the complete sets of mutually exclusive topological relationships for all type combinations. Completeness and mutual exclusion are shown by a proof technique called *proof-by-constraint-and-drawing*. Due to the resulting large numbers of predicates and the difficulty of handling them, the user is provided with the concepts of *topological cluster predicates* and *topological predicate clusters*, which permit to reduce the number of predicates to be dealt with in a user-defined and/or application-specific manner.

Key Words: Topological predicate, topological constraint rule, topological cluster predicate, topological predicate cluster, complex spatial data type, proof-by-constraint-and-drawing, 9-intersection model

1 Introduction

In recent years, the exploration of topological relationships between objects in space has turned out to be a multi-disciplinary research issue involving disciplines like artificial intelligence, CAD/CAM systems, cognitive science, computer vision, geographical information science, image databases, linguistics, psychology, robotics, spatial analysis, spatial database systems, and spatial reasoning. From a database and GIS perspective, their development has been motivated by the necessity of formally defined topological predicates as filter conditions for spatial selections and spatial joins in spatial query languages and as a support for spatial data retrieval and analysis tasks, both at the user definition level for reasons of conceptual clarity and at the query processing level for reasons of efficiency.

Topological relationships like *overlap*, *inside*, or *meet* describe purely qualitative properties that characterize the relative positions of spatial objects and are preserved under continuous transformations such as translation, rotation, and scaling. They deliberately exclude any consideration of quantitative, metric measures like distance or direction measures and are associated with notions like adjacency, coincidence, connectivity, inclusion, and continuity. Two important approaches for the definition of topological relationships are the 9-*intersection model* [13], which rests on point set theory and point set topology and which we will take as the foundation of our paper, as well as the *RCC model* [8], which employs spatial logic. Both models are essentially tailored to the treatment of simplified abstractions of spatial objects like simple lines and simple regions. Simple lines are one-dimensional, continuous features embedded in the plane with two end points, and simple regions are two-dimensional point sets topologically equivalent to a closed disc. Single points have not found much interest, since their interrelations are trivial.

Unfortunately, these simple geometric structures are inadequate abstractions for real spatial applications since they are insufficient to cope with the variety and complexity of geographic reality. Thus, universal and versatile type specifications are needed for (more) complex spatial objects that are usable in many different applications. With regard to *complex points*, we allow finite collections of single points as point objects (e.g., to gather the positions of all lighthouses in the U.S.). With regard to *complex lines*, we permit arbitrary, finite collections of one-dimensional curves, i.e., spatially embedded networks possibly consisting of several disjoint connected components, as line objects (e.g., to model the ramifications of the Nile Delta). With regard to *complex regions*, the two main extensions relate to separations of the exterior (holes) and to separations of the interior (multiple components). For example, countries (like Italy) can be made up of multiple components (like the mainland and the offshore islands) and can have holes (like the Vatican). From a formal point of view, spatial data types should be closed under the geometric set operations *union*, *intersection*, and *difference*. This is not the case for the simple types and means that such an operation applied to two point, two line, and two region objects, respectively, must yield a well-defined object and may not leave the corresponding type definition. Hence, a first goal of this paper is to introduce and formalize spatial data types for complex points, complex lines and complex regions. The formal basis is provided by point set theory and point set topology.

With the increasing integration of *complex spatial data types* into GIS and into spatial extension packages of commercial database systems, an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. Hence, a second goal of this paper is to explore and derive the possible topological relationships between all combinations of complex spatial data types on the basis of the well known 9-intersection model. For this purpose, we draw up collections of constraints specifying conditions for valid topological relationships and satisfying the properties of *completeness* and *exclusiveness*. The property of completeness ensures a full coverage of all topological situations. The property of exclusiveness ensures that two different relationships cannot hold for the same two spatial objects. Topological relationships between complex spatial objects are also interesting from a simply theoretical point of view for the aforementioned disciplines and are important for an extension of spatio-temporal predicates [20] from simple to complex spatio-temporal objects (moving objects) [27].

The resulting large numbers of predicates for each type combination make them difficult to handle for the user. Hence, a third goal of this paper is to provide the user with the concepts of *topological cluster predicates* and *topological predicate clusters*. These permit to reduce the number of predicates to be dealt with in a user-defined and/or application-specific manner.

The remainder of the paper is organized as follows: Section 2 discusses related work regarding spatial objects and topological relationships. Section 3 formalizes the spatial data model for which topological relationships will be investigated. Section 4 explains the general strategy for deriving topological relationships from the 9-intersection model. In Section 5 all topological relationships between spatial objects of the same type are analyzed. Section 6 deals with topological relationships between spatial objects of different types. Section 7 introduces the concepts of topological cluster predicates and topological predicate clusters and shows their possible specification by means of the DDL of a query language. Finally, Section 8 draws some conclusions and discusses future work.

2 Related Work

In this section we discuss related work about spatial objects as the operands of topological relationships (predicates) (Section 2.1) and about topological relationships themselves (Section 2.2).

2.1 Spatial Objects

In the past, numerous data models and query languages for spatial data have been proposed with the aim of formulating and processing spatial queries in databases and GIS (e.g., [12, 24, 26, 32, 33, 34]). *Spatial data types* (see [34] for a survey) like *point*, *line*, or *region* are the central concept of these approaches. They provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. Topological predicates operate on instances of these data types, called *spatial objects*. So far, only simple object structures like single points, continuous lines, and simple regions have been specified and used as arguments of topological predicates.

Only a few models [4, 26, 29, 30] have been developed towards complex spatial objects. The approach in [4] allows self-intersecting and self-touching lines, and the object representations resulting from its definitions are not necessarily unique. The definitions of complex spatial data types in [26] are based on a finite geometric domain (so-called realm) whereas we define our data types in the infinite Euclidean plane. From a structural perspective, they are quite similar to ours so that they can be used as an implementation of our model. The OpenGIS Consortium (OGC) has incorporated similar generalized geometric structures, called *simple features*, into their OGC Abstract Specification [29] and into the Geography Markup Language (GML) [30], which is an XML encoding for the transport and storage of geographic information. These geometric structures are only described informally and called *MultiPoint*, *MultiLineString*, and *MultiPolygon*. In contrast to the other approaches, different components of the same spatial object may overlap. Database vendors have integrated complex spatial data types into their spatial extension packages through extensibility mechanisms. Examples are the Informix Geodetic DataBlade [28], the Oracle Spatial Cartridge [31], and DB2's Spatial Extender [9]. All of them more or less borrow their technology from ESRI's Spatial Database Engine (SDE) [21].

2.2 Topological Relationships

Topological predicates between spatial objects in two-dimensional space belong to the most investigated topics of spatial data modeling and reasoning. Their importance and necessity has already been recognized for a long time [22, 1]. An important approach for characterizing them rests on the so-called 9-*intersection model* [16] as an extension and generalization of the original 4-*intersection model* [13, 10, 15]. Both models use point sets and point set topology as their formal framework. Based on the 9-intersection model, a complete collection of mutually exclusive topological relationships can be determined for each combination of simple spatial data types. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object A with the corresponding components of another object B.[1] Each intersection is tested with regard to the topologically invariant criteria of emptiness and non-emptiness. The topological relationship between two spatial objects A and B can be expressed by evaluating the matrix in Table 1.

For this matrix $2^9 = 512$ different configurations are possible from which only a certain subset makes sense depending on the *definition* and *combination* of spatial objects just considered. For each combination of spatial types this means that each of its predicates is associated with a unique intersection matrix

[1]The 4-intersection model only considers the interior and the boundary of each spatial object.

$$\begin{pmatrix} A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ \partial A \cap B^\circ \neq \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^- \cap B^\circ \neq \varnothing & A^- \cap \partial B \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

Table 1: The 9-intersection matrix.

so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness. Topological relationships that have been investigated so far are restricted in the sense that their argument objects are not allowed to have a very general structure. It is just the objective of this paper to give very general and versatile definitions of spatial objects and to identify the topological relationships between them.

Topological relationships have been first investigated for simple regions [13, 10, 14, 15, 7]. For two simple regions eight meaningful (out of 512 possible) configurations have been identified which lead to the well known eight predicates called *disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, and *coveredBy* (Table 2).

$$\underset{\textit{disjoint}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{meet}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{overlap}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{equal}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{\textit{inside}}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{contains}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\textit{covers}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\textit{coveredBy}}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

Table 2: The eight topological predicates for two simple regions.

The 4-intersection / 9-intersection model on simple regions has been extended and refined in a number of ways especially with further topological invariants (like the dimension of the intersection components, their types (touching, crossing), the number of components) to discover more details about topological relationships (e.g., [7, 19]). A generalization of the aforementioned eight topological predicates to complex regions can be found in [35]; their implementation is described in [36].

It is surprising that topological predicates on complex regions have so far not been defined. But the definition of these predicates is particularly important for spatial query languages that aim at integrating complex regions having holes and separations. Three works have so far contributed to a definition of topological relationships for more complex regions. In [6] the so-called TRCR (Topological Relationships for Composite Regions) model only allows sets of disjoint simple regions without holes. Topological relationships between these composite regions are defined in an ad hoc manner and are not systematically derived from the underlying model. Further, the model is only related to but not directly based on the 9-intersection model. In [18] topological relationships of simple regions with holes are considered. Unfortunately, multi-part regions are not permitted. While the authors take into account the number of components (areas without holes, holes) of two regions and the large number of topological relationships between all component pairs of both regions, we pursue a global approach that is independent of the number of components. Hence, a further goal of this paper is to provide an integrated treatment of holes and separations for regions and to define topological predicates on complex regions in a systematic way. Our work in [2] already gives the rough idea of our mechanism for deriving topological relationships for complex regions.

For two simple lines 33 topological relationships [16, 11, 5] can be found. Additional 24 relationships exist for more complex lines which are allowed to be a connected graph without loops and without self-intersections [16]. We generalize this approach in the sense that we also permit complex lines to have loops and to consist of several graph components. Topological predicates between simple points are trivial: either two simple points are *disjoint* or they are *equal*. We will consider predicates between complex points as finite collections of single points.

A simple point can be located *on* one of the endpoints of a simple line, *in* the interior of a simple line, or be *disjoint* from a simple line. We are interested in the relationships between complex points and complex lines. For a simple point and a simple region we obtain the three predicates *disjoint*, *meet*, and *inside*. We will identify the relationships between complex points and complex regions. For a simple line and a simple region 19 topological relationships [17] can be distinguished. Here we are interested in the nature and number of topological relationships between a complex line and a complex region. Table 3 summarizes the results obtained so far.

	simple point	simple line	simple region
simple point	2	3	3
simple line	3	33	19
simple region	3	19	8

Table 3: Numbers of topological predicates between two simple spatial objects.

Table 3 also indicates that the numbers of topological predicates can already become quite large for topological predicates between simple spatial objects so that the predicates are difficult to handle by the user. A further increase of the number of predicates has even to be expected for complex spatial objects. The calculus-based method in [7] groups all its possible cases into a few meaningful topological relationships. The grouping is performed on the basis of the emptiness and non-emptiness of component intersections, inclusion and non-inclusion of one object in another object, and the dimension of component intersections. In contrast to this, our concept will exclusively rest on the emptiness and non-emptiness of component intersections.

3 Complex Spatial Objects

This section defines the underlying spatial data model for our topological predicates. We strive for a very general, abstract definition of complex spatial objects (see Figure 1) in the Euclidean plane $\mathbb{R}^2$. The task is to determine those point sets that are admissible for complex point (Section 3.1), complex line (Section 3.2), and complex region (Section 3.3) objects. For complex lines and complex regions, we give both an “unstructured” and a “structured” definition; for complex points both definitions coincide. The unstructured definition purely determines the point set of a line or region. The structured definition gives a unique representation and emphasizes the component view of a spatial object. A complex point may include several points, a complex line may be a spatially embedded network possibly consisting of several components, and a complex region may be a multi-part region possibly with holes. The formal framework purely employs point set theory and point set topology [23] since we disregard shape aspects and metric properties and focus on the study of topological relationships. For each spatial data type we specify the topological notions of *boundary*, *interior*, *exterior*, and *closure* since these notions are later needed for the specification of topological relationships.

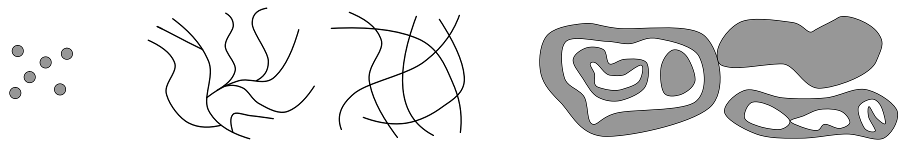


Figure 1: Examples of a complex point object (a), a complex line object (b), and a complex region object (c).

3.1 Complex Points

A value of type *point* is defined as a finite set of isolated points in the plane. Finiteness of the number of components is required in all parts of the spatial data model since we are only able to handle finite collections of spatial objects in geographic applications.

Definition 3.1 The spatial data type *point* is defined as

$$point = \{P \subset \mathbb{R}^2 \,|\, P \text{ is finite}\}$$

We call a value of this type *complex point*. If $P \in point$ is a singleton set, i.e., $|P| = 1$, P is denoted as a *simple point*. □

In other words, type *point* (Figure 1a) contains all finite sets of the power set of $\mathbb{R}^2$. In particular, the empty set, which is the identity of geometric union, is admitted since it can be the result of a geometric operation, e.g., if a point object has nothing in common with another point object in a geometric intersection operation.

Since we intend to later apply the 9-intersection model to complex points, we have to give a definition for the topological notions of boundary, interior, and exterior of a complex point. For a simple point p we specify $\partial p = \varnothing$ and $p^\circ = p$, which is the commonly accepted definition. For a complex point $P = \{p_1, \ldots, p_n\}$ we then obtain $\partial P = \varnothing$, $P^\circ = \bigcup_{i=1}^{n} p_i^\circ$, and $P^- = \mathbb{R}^2 - (\partial P \cup P^\circ) = \mathbb{R}^2 - P^\circ$. The closure $\overline{P}$ is given as $\overline{P} = \partial P \cup P^\circ = P^\circ$.

3.2 Complex Lines

Before we start with a definition for complex lines (Figure 1b), we need a few definitions of some well-known and needed topological concepts. We assume the existence of the Euclidean distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $d(p,q) = d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$. With the notion of distance, we can now proceed to define what is meant by a *neighborhood* of a point in $\mathbb{R}^2$.

Definition 3.2 Let $q \in \mathbb{R}^2$ and $\varepsilon \in \mathbb{R}^+$. The set $N_\varepsilon(q) = \{p \in \mathbb{R}^2 \,|\, d(p,q) < \varepsilon\}$ is called the *open neighborhood of radius* ε *and center* q. The set $\overline{N}_\varepsilon(q) = \{p \in \mathbb{R}^2 \,|\, d(p,q) \leq \varepsilon\}$ is called the *closed neighborhood of radius* ε *and center* q. Any open (closed) neighborhood with center q is denoted by $N(q)$ ($\overline{N}(q)$). Any neighborhood (closed or open) with center q is denoted by $N^*(q)$. □

We are now able to define the notion of a *continuous mapping* which preserves neighborhood relations between mapped points in two spaces of the plane. Hence, the property of continuity of this mapping ensures the maintenance of the closure and connectivity of the mapping domain for its image. These mappings are also called *topological transformations* and include translation, rotation, and scaling.

Definition 3.3 Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}^2$. Then f is said to be *continuous at a point* $x_0 \in X$ if, given an arbitrary number $\varepsilon > 0$, there exists a number $\delta > 0$ (usually depending on ε) such that for every $x \in N_\delta(x_0) \cap X$ we obtain that $f(x) \in N_\varepsilon(f(x_0))$. The mapping f is said to be *continuous on* X if it is continuous at every point of X. □

For a function $f : X \to Y$ and a set $A \subseteq X$ we introduce the notation $f(A) = \{f(x) \,|\, x \in A\}$. Definition 3.3 already enables us now to give an unstructured definition for complex lines as the union of the images of a finite number of continuous mappings.

Definition 3.4 The spatial data type *line* is defined as

$$\begin{aligned} line = \{L \subset \mathbb{R}^2 \,|\, & \text{(i)} \;\; L = \textstyle\bigcup_{i=1}^n f_i([0,1]) \text{ with } n \in \mathbb{N}_0 \\ & \text{(ii)} \;\; \forall 1 \le i \le n : f_i : [0,1] \to \mathbb{R}^2 \text{ is a continuous mapping} \\ & \text{(iii)} \; \forall 1 \le i \le n : |f_i([0,1])| > 1\} \end{aligned}$$

We call a value of this type *complex line*. □

The first condition also allows a line object to be the empty set ($n = 0$ in Definition 3.4). The third condition avoids degenerate line objects consisting only of a single point.

For a structured definition, we separate the point set of a complex line into appropriate components. We first consider a *single-component line* as the image of a single continuous mapping and distinguish several special cases.

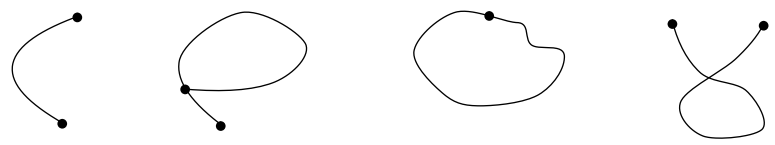
Definition 3.5 Let $l \subset \mathbb{R}^2$, $l \neq \varnothing$.

(i)	l is a *single-component line*	$\Leftrightarrow$	$l = f([0,1])$ where $f : [0,1] \to \mathbb{R}^2$ is a continuous mapping and $\|f([0,1])\| > 1$
(ii)	l is a *simple line*	$\Leftrightarrow$	l is a single-component line $\wedge$ $\forall a,b \in [0,1], a \neq b : f(a) \neq f(b)$
(iii)	l is a *self-touching line*	$\Leftrightarrow$	l is a single-component line $\wedge$ $\exists a \in \{0,1\}\ \exists b \in \,]0,1[\, : f(a) = f(b)$
(iv)	l is a *closed line*	$\Leftrightarrow$	l is a single-component line $\wedge$ $f(0) = f(1)$
(v)	l is a *self-intersecting line*	$\Leftrightarrow$	l is a single-component line $\wedge$ $\exists a,b \in \,]0,1[, a \neq b : f(a) = f(b)$
(vi)	l is a *non-self-intersecting line*	$\Leftrightarrow$	l is a single-component line $\wedge$ $\forall a,b \in \,]0,1[, a \neq b : f(a) \neq f(b)$ $\wedge$ $\forall a \in \{0,1\}\ \forall b \in \,]0,1[\, : f(a) \neq f(b)$

The values $f(0)$ and $f(1)$ are called the *end points* of l. □

Intuitively, a single-component line models all curves that can be drawn on a sheet of paper from a starting point to an end point without lifting the pen. Note that Definitions 3.5 (ii) to (vi) are not mutually exclusive. For instance, a single-component line can be self-intersecting and closed. Figure 2 shows some examples.

For a unique definition of complex lines, only non-self-intersecting lines are adequate since the other line types in Definition 3.5 lead to modeling or uniqueness problems. Obviously, self-touching and closed lines are too specialized. Simple lines are also too limited because they do not allow closed lines. Self-intersecting and single-component lines are too general because they allow multiple representations. The self-intersecting line in Figure 2d can, e.g., also be represented by three continuous mappings, namely



(a) (b) (c) (d)

Figure 2: A single-component line as a simple line (a), as a self-touching line (b), as a closed line (c), and as a self-intersecting line (d). The thick points indicate the end points of the different kinds of lines.

one mapping for the closed part and one mapping for each remaining part. Non-self-intersecting lines, also denoted as *curves*, cover simple and closed lines and can serve as a "canonical" representation.

Definition 3.6 Let S be the set of all non-self-intersecting lines over $\mathbb{R}^2$. Two lines $l_1, l_2 \in S$ with corresponding continuous mappings f_1 and f_2 are called *quasi-disjoint* if, and only if, $(\forall a, b \in]0,1[: f_1(a) \neq f_2(b)) \wedge \neg((f_1(0) = f_2(0) \wedge f_1(1) = f_2(1)) \vee (f_1(0) = f_2(1) \wedge f_1(1) = f_2(0)))$. They *meet* in an end point p if, and only if, they are quasi-disjoint and $\exists a, b \in \{0,1\} : f_1(a) = p = f_2(b)$. □

Note that due to the uniqueness constraint the definition of *quasi-disjoint* forbids that two non-self-intersecting lines form a loop.

Next, we introduce the concept of a *block*, which is used in Definition 3.8 to specify a connected component of a complex line. Such a connected component is represented as a spatially embedded planar graph, in which, again due to the uniqueness constraint, an end point is either the end point of only a single non-self-intersecting line or shared by more than two non-self-intersecting lines. Intuitively, this definition means that in Figure 1b we have to draw an explicit end point at each intersection point.

Definition 3.7 The set B of *blocks* over S is defined as

$$
\begin{aligned}
B = \{\textstyle\bigcup_{i=1}^{m} l_i \,|\, & \text{(i)} \;\; m \in \mathbb{N}, \forall 1 \leq i \leq m : l_i \in S \\
& \text{(ii)} \;\; \forall 1 \leq i < j \leq m : l_i \text{ and } l_j \text{ are quasi-disjoint} \\
& \text{(iii)} \;\; m > 1 \Rightarrow \forall 1 \leq i \leq m \; \exists 1 \leq j \leq m, \; i \neq j : l_i \text{ and } l_j \text{ meet} \\
& \text{(iv)} \;\; \forall p \in \textstyle\bigcup_{i=1}^{m} \{f_i(0), f_i(1)\} : card(\{f_i \,|\, 1 \leq i \leq m \wedge f_i(0) = p \wedge f_i(1) = p\}) \neq 0 \\
& \qquad \vee \; card(\{f_i \,|\, 1 \leq i \leq m \wedge (f_i(0) = p \wedge f_i(1) \neq p) \vee \\
& \qquad\qquad (f_i(0) \neq p \wedge f_i(1) = p)\}) \neq 2
\end{aligned}
$$

Two blocks $b_1, b_2 \in B$ are *disjoint* if, and only if, $b_1 \cap b_2 = \varnothing$. □

Condition (iii) ensures uniqueness of representation and disallows end points which have exactly two different emanating segments. We are now able to give the structured definition for complex lines.

Definition 3.8 The spatial data type *line* is defined as

$$
\begin{aligned}
line = \{\textstyle\bigcup_{i=1}^{n} b_i \,|\, & \text{(i)} \;\; n \in \mathbb{N}_0, \forall 1 \leq i \leq n : b_i \in B \\
& \text{(ii)} \; \forall 1 \leq i < j \leq n : b_i \text{ and } b_j \text{ are disjoint}\}
\end{aligned}
$$

We call a value of this type *complex line*. □

A line object is the empty object (empty set) if $n = 0$ in Definition 3.8. Figure 3 shows a complex line consisting of six continuous mappings according to the structured view. The boundary of a complex line L is the set of its end points minus those end points that are shared by several non-self-intersecting lines. The shared points belong to the interior of a complex line. Based on Definition 3.4, let $E(L) = \bigcup_{i=1}^{n} \{f_i(0), f_i(1)\}$ be the set of end points of all non-self-intersecting lines. We obtain

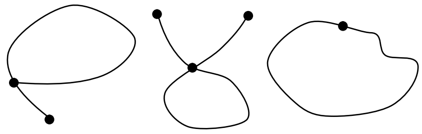


Figure 3: A complex line. The thick points indicate end points of non-self-intersecting lines.

$$\partial L = E(L) - \{p \in E(L) \,|\, card(\{f_i \,|\, 1 \leq i \leq m \wedge f_i(0) = p\}) + card(\{f_i \,|\, 1 \leq i \leq m \wedge f_i(1) = p\}) \neq 1\}$$

Let $L \neq \varnothing$. It is possible that ∂L is empty. The closure $\overline{L}$ of L is the set of all points of L including the end points. Therefore $\overline{L} = L$ holds. For the interior of L we obtain $L^\circ = \overline{L} - \partial L = L - \partial L \neq \varnothing$, and for the exterior we get $L^- = \mathbb{R}^2 - L$, since $\mathbb{R}^2$ is the embedding space.

3.3 Complex Regions

Our definition of complex regions is based on point set theory and point set topology [23]. Regions are embedded into the two-dimensional Euclidean space $\mathbb{R}^2$ and modeled as special infinite point sets. We briefly introduce some needed concepts from point set topology in $\mathbb{R}^2$.

Definition 3.9 Let $X \subseteq \mathbb{R}^2$ and $q \in \mathbb{R}^2$. q is an *interior point* of X if there exists a neighborhood N^* such that $N^*(q) \subseteq X$. q is an *exterior point* of X if there exists a neighborhood N^* such that $N^*(q) \cap X = \varnothing$. q is a *boundary point* of X if q is neither an interior nor exterior point of X. q is a *closure point* of X if q is either an interior or boundary point of X.

The set of all interior points of X is called the *interior* of X and is denoted by X°. The set of all exterior points of X is called the *exterior* of X and is denoted by X^-. The set of all boundary points of X is called the *boundary* of X and is denoted by ∂X. The set of all closure points of X is called the *closure* of X and is denoted by $\overline{X}$.

A point q is a *limit point* of X if for every neighborhood $N^*(q)$ holds that $(N^* - \{q\}) \cap X \neq \varnothing$. X is called an *open set* in $\mathbb{R}^2$ if $X = X^\circ$. X is called a *closed set* in $\mathbb{R}^2$ if every limit point of X is a point of X. □

It follows from the definition that every interior point of X is a limit point of X. Thus, limit points need not be boundary points. The converse is also true. A boundary point of X need not be a limit point; it is then called an *isolated* point of X. For the closure of X we obtain that $\overline{X} = \partial X \cup X^\circ$.

It is obvious that arbitrary point sets do not necessarily form a region. But open and closed point sets in $\mathbb{R}^2$ are also inadequate models for complex regions since they can suffer from undesired geometric anomalies (Figure 4). A complex region defined as an open point set runs into the problem that it may have missing lines and points in the form of cuts and punctures. At any rate, its boundary is missing. A complex region defined as a closed point set admits isolated or dangling point and line features. *Regular closed* point sets [37] avoid these anomalies.

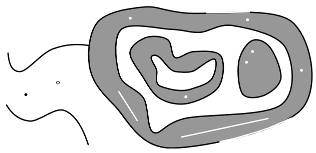


Figure 4: Examples of possible geometric anomalies of a region object.

Definition 3.10 Let $X \subseteq \mathbb{R}^2$. X is called *regular closed* if, and only if, $X = \overline{X^\circ}$. □

The effect of the *interior* operation is to eliminate dangling points, dangling lines, and boundary parts. The effect of the *closure* operation is to eliminate cuts and punctures by appropriately supplementing points and to add the boundary. Closed neighborhoods (Definition 3.2), e.g., are regular closed sets.

For the specification of the *region* data type, definitions are needed for bounded and connected sets.

Definition 3.11 (i) Two sets $X, Y \subseteq \mathbb{R}^2$ are said to be *separated* if, and only if, $X \cap \overline{Y} = \varnothing = \overline{X} \cap Y$. A set $X \subseteq \mathbb{R}^2$ is *connected* if, and only if, it is not the union of two non-empty separated sets. (ii) Let $q = (x,y) \in \mathbb{R}^2$. Then the *length* or *norm* of q is defined as $||q|| = \sqrt{x^2+y^2}$. (iii) A set $X \subseteq \mathbb{R}^2$ is said to be *bounded* if there exists a number $r \in \mathbb{R}^+$ such that $||q|| < r$ for every $q \in X$. □

We are now able to give an unstructured type definition for complex regions:

Definition 3.12 The spatial data type *region* is defined as

$$\begin{aligned} \mathit{region} = \{R \subset \mathbb{R}^2 \,|\, & \text{(i)} \;\; R \text{ is regular closed} \\ & \text{(ii)} \;\; R \text{ is bounded} \\ & \text{(iii)} \text{ The number of connected sets of } R \text{ is finite}\} \end{aligned}$$

We call a value of this type *complex region*. □

A region object can also be the empty object (empty set). In fact, this very "unstructured" definition models complex regions possibly consisting of several components and possibly having holes. But since the topological predicates of the 9-intersection model only work on simpler regions, we have to take a more fine-grained and structured view of regions. The structured definition of type *region* distinguishes simple regions, simple regions with holes, and complex regions.

Definition 3.13 A *simple region* is a bounded, regular closed set homeomorphic (i.e., topologically equivalent) to a closed neighborhood in $\mathbb{R}^2$. □

This, in particular, means that a simple region has a connected interior, a connected boundary, and a single connected exterior. Hence, it does not consist of several components, and it does not have holes.

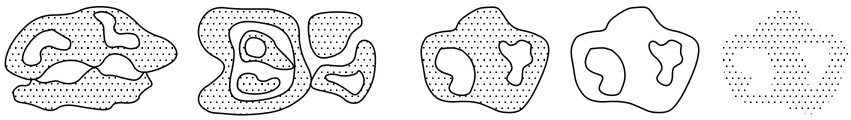
The concept of a hole is topologically not directly inferable since point set topology does not distinguish between "outer" exterior and "inner" exterior of a set. This requires an explicit and constructive definition of a region containing holes and a use of the topological predicates *meet*, *covers*, *coveredBy*, *contains*, and *disjoint* for simple regions, as they are defined by the 9-intersection model in Table 2.

Definition 3.14 Let $\{F_0, \ldots, F_n\}$ be a set of simple regions, and let $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}, k, n \in \mathbb{N}, k \le n$, be a total, injective mapping. The regular set $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ is called a *simple region with holes* or a *face*, and $F_1, \ldots, F_n$ are called *holes* if, and only if,

$$\begin{aligned} &\text{(i)} && \forall 1 \le i \le n : \mathit{contains}(F_0, F_i) \vee (\mathit{covers}(F_0, F_i) \wedge |F_0 \cap F_i| = 1) \\ &\text{(ii)} && \forall 1 \le i < j \le n : \mathit{disjoint}(F_i, F_j) \vee (\mathit{meet}(F_i, F_j) \wedge |F_i \cap F_j| = 1) \\ &\text{(iii)} && \nexists \{\pi(1), \ldots, \pi(k)\} \subseteq \{1, \ldots, n\} : \mathit{covers}(F_0, F_{\pi(1)}) \wedge \\ &&& \mathit{meet}(F_{\pi(1)}, F_{\pi(2)}) \wedge \cdots \wedge \mathit{meet}(F_{\pi(k-1)}, F_{\pi(k)}) \wedge \mathit{coveredBy}(F_{\pi(k)}, F_0) \\ &\text{(iv)} && \nexists \{\pi(1), \ldots, \pi(k)\} \subseteq \{1, \ldots, n\} : \mathit{meet}(F_{\pi(1)}, F_{\pi(2)}) \wedge \\ &&& \mathit{meet}(F_{\pi(2)}, F_{\pi(3)}) \wedge \cdots \wedge \mathit{meet}(F_{\pi(k-1)}, F_{\pi(k)}) \wedge \mathit{meet}(F_{\pi(k)}, F_{\pi(1)}) \end{aligned}$$

Let *srh* be the set of all simple regions with holes. □

Figure 5c gives an example of a face. The first two conditions allow a hole within a face to touch the boundary of F_0 or of another hole in at most a single point. This is necessary in order to achieve closure under the geometric operations *union*, *intersection*, and *difference* (see also [25, 34]). For example, a (regularized) subtraction of a face A from a face B may lead to such a hole in B. On the other hand, to allow two holes to have a partially common border makes no sense because then adjacent holes could be merged to a single hole by eliminating the common border (similarly for adjacency of a hole with the boundary of F_0). The third condition prevents the formation of "open hole chains" where any two subsequent holes meet and both the first and the last hole touch F_0. The fourth condition prevents the formation of "closed hole chains" within the face where any two subsequent holes meet and both the first and the last hole meet. All four conditions together ensure uniqueness of representation, i.e., there are no two different interpretations of the point set describing a face. Hence, a face is atomic and cannot be decomposed into two or more faces. For example, the configuration shown in Figure 5a must be interpreted as two faces with two holes and not as a single face with four holes.



(a) (b) (c) (d) (e)

Figure 5: Unique representation of a face (a), a complex region with five faces (b), a simple region with two holes (c), its boundary (d), and its interior (e).

Let $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ be a simple region with holes $F_1, \ldots, F_n$. Then the boundary of F (Figure 5d) is given as $\partial F = \bigcup_{i=0}^{n} \partial F_i$, and the interior of F (Figure 5e) is defined as $F^\circ = F_0^\circ - \bigcup_{i=1}^{n} F_i$.

We are now able to give the structured definition for complex regions.

Definition 3.15 The spatial data type *region* is defined as

$$\begin{aligned} \mathit{region} = \{R \subseteq \mathbb{R}^2 \,|\, & \text{(i)} \quad R = \textstyle\bigcup_{i=1}^{n} F_i \text{ with } n \in \mathbb{N}_0 \\ & \text{(ii)} \ \ \forall 1 \le i \le n : F_i \in \mathit{srh} \\ & \text{(iii)} \ \forall 1 \le i < j \le n : F_i^\circ \cap F_j^\circ = \varnothing \\ & \text{(iv)} \ \forall 1 \le i < j \le n : \partial F_i \cap \partial F_j = \varnothing \ \vee \ |\partial F_i \cap \partial F_j| \text{ is finite}\} \end{aligned}$$

We call a value of this type *complex region*. □

A region object is the empty object (empty set) if $n = 0$ in Definition 3.15. Figure 5b shows an example of a region with five faces. The definition requires of a face to be disjoint to another face, or to meet another face in one or several single boundary points, or to lie within a hole of another face and possibly share one or several single boundary points with the boundary of the hole. Faces having common connected boundary parts with other faces or holes are disallowed. The argumentation is similar to that for the face definition.

Let $F = \bigcup_{i=1}^{n} F_i$ be a non-empty region with faces $\{F_1, \ldots, F_n\}$. Then the boundary of F is given as $\partial F = \bigcup_{i=1}^{n} \partial F_i \ (\neq \varnothing)$, and the interior of F is given as $F^\circ = \bigcup_{i=1}^{n} F_i^\circ = F - \partial F \ (\neq \varnothing)$. Further, we obtain $\overline{F} = \partial F \cup F^\circ = F$ and $F^- = \mathbb{R}^2 - \overline{F} = \mathbb{R}^2 - F \ (\neq \varnothing)$.

4 Deriving Topological Relationships from the 9-Intersection Model

The preceding section has provided structured and unstructured views of three kinds of complex spatial objects. An apparently promising approach to deriving topological relationships from them is to leverage the structured view of a spatial object. But the following expositions reveal that considering components leads to rather complicated and impractical models.

We demonstrate this by first considering two simple regions A and B with n and m holes, respectively. If we take into account the regions A and B *without* holes and call them A^* and B^*, respectively, the total number of topological relationships that can be specified between A^* and its holes with B^* and its holes amounts to $(n+m+2)^2$ [18]. It has also been shown in [18] that this number can be reduced to $mn+m+n+1$. The problems of this approach are the dependency on the number of holes and the resulting large number of topological relationships.

We are confronted with a similar problem if we take another strategy and have a look on the topological relationships between two complex regions A and B with n and m faces, respectively, possibly with holes. Each face of A is in relationship with any face of B. This gives a total of $8^{n \cdot m}$ possible topological configurations if we take the eight topological relationships between two simple regions with holes, as they are specified in [35], as the basis. As a result, the total number of relationships between the faces of two complex regions depends on the numbers of faces, is therefore not bounded by a constant, and increases in an exponential way. This approach is obviously not manageable and thus not acceptable.

Hence, the comparison of structural elements of the objects with respect to their topological relationships does not seem to be an adequate and efficient method. On the contrary, this detailed investigation is usually not desired and thus even unnecessary. For instance, if two regions intersect (according to some definition), the number of intersecting face pairs, as long as it is greater than 0, is irrelevant since it does not influence the fact of intersection. Consequently, the analysis of topological relationships between two complex spatial objects requires a more general strategy.

Our strategy for the analysis of topological relationships between two complex spatial objects is simple and yet very general and expressive. Instead of applying the 9-intersection model to point sets belonging to simple spatial objects, we extend it to point sets belonging to complex spatial objects. Due to the special features of the objects (point, linear, areal properties), the embedding space (here: $\mathbb{R}^2$), the relation between the objects and the embedding space (e.g., it makes a difference whether we consider a point in $\mathbb{R}$ or in $\mathbb{R}^2$), and the employed spatial data model (e.g., discrete, continuous), a number of topological configurations cannot exist and have to be excluded. For each pair of complex spatial data types, our goal is to determine topological constraints or conditions that have to be satisfied. These serve as criteria for excluding all impossible configurations. The approach taken employs a proof technique called *Proof-By-Constraint-And-Drawing*. It starts with the 512 possible matrices and is a two-step process:

(i) For each type combination we give the formalization of a collection of topological *constraint* rules for existing relationships in terms of the nine intersections. For each constraint rule we give reasons for its validity, correctness, and meaningfulness. The evaluation of each constraint rule gradually reduces the set of the currently valid matrices by all those matrices not fulfilling the constraint rule under consideration.

(ii) The existence of topological relationships given by the remaining matrices is verified by realizing prototypical spatial configurations in $\mathbb{R}^2$, i.e., these configurations can be *drawn* in the plane.

Still open issues relate to the evaluation order, completeness, and minimality of the collection of constraint rules. Each constraint rule is a predicate that is matched with all intersection matrices under consideration. All constraint rules must be satisfied together so that they represent a conjunction of predicates. To say it in other words, constraint rules are all formulated in conjunctive normal form. Since the conjunction (logical *and*) operator is commutative and associative, the *evaluation order* of the constraint rules is irrelevant; the final result is always the same.

The *completeness* of the collection of constraints is directly ensured by the second step of the two-step process provided that all spatial configurations for the remaining matrices can be drawn.

The aspect of *minimality* addresses the possible redundancy of constraint rules. Redundancy can arise for two reasons. First, several constraint rules may be correlated in the sense that one of them is more general than the others, i.e., it eliminates at least the matrices excluded by all the other, covered constraints. This can be easily checked by analyzing the constraint rules themselves and searching for the most non-restrictive and common constraint rule. Even then the same matrix can be excluded by several constraint rules simultaneously. Second, a constraint rule can be covered by some combination of other constraint rules. This can be checked by a comparison of the matrix collection fulfilling all n constraint rules with the matrix collection fulfilling $n-1$ constraint rules. If both collections are equal, then the omitted constraint rule was implied by the combination of the other constraint rules and is therefore redundant. In this paper, we are not so much interested in the aspect of minimality since our goal is to identify the topologically invalid intersection matrices. We are willing to accept a certain (but small) degree of redundancy.

5 Topological Relationships between Two Complex Spatial Objects of Equal Type

First we analyze the topological relationships between two complex spatial objects which have the same type and thus share the same properties and dimension. This leads to the three type combinations *point*/*point* (Section 5.1), *line*/*line* (Section 5.2), and *region*/*region* (Section 5.3).

5.1 Topological Relationships between Two Complex Points

Topological relationships on complex point objects have so far not been explored in the literature, since the assumption has always been that a point object only consists of a single point and since the topological relationships between two single points are rather trivial (either disjointedness or equality). In Section 2.1 we have motivated why more complex spatial objects are needed in general. For example, consider an operation $crossing : line \times line \to point$ taking two line objects as operands, computing all single intersection points (common linear parts are ignored here), and collecting them in a single *point* object. This shows the necessity to deal with topological predicates on *complex* point objects.

We now present the constraint rules for two complex point objects A and B defined according to Section 3.1. Each constraint rule is first formulated colloquially and afterwards formalized by employing the nine intersections. Then a rationale is given explaining why the constraint rule makes sense and is correct. We presuppose that A and B are not empty, because topological relationships for empty operands are not meaningful.

Lemma 5.1.1 All intersections comprising an operand with a boundary operator yield the empty set, i.e.,

$$\forall C \in \{A^\circ, \partial A, A^-\} : C \cap \partial B = \varnothing \;\wedge\; \forall D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing$$

Proof. According to the definition of a complex point $\partial A = \partial B = \varnothing$ holds. Hence, the intersection of the empty set with any other component yields the empty set. □

Lemma 5.1.2 The exteriors of two complex point objects always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. We know that $A \cup A^- = \mathbb{R}^2$ and $B \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $A = \mathbb{R}^2$, or (ii) $B = \mathbb{R}^2$, or (iii) $A \cup B = \mathbb{R}^2$. All three situations are impossible, since A, B, and $A \cup B$ are finite sets and $\mathbb{R}^2$ is an infinite set. Thus $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$, and $A \cup B \subset \mathbb{R}^2$ holds, and $(\mathbb{R}^2 - A) \cap (\mathbb{R}^2 - B) \neq \varnothing$. □

Lemma 5.1.3 Each non-empty part of a complex point intersects at least one non-empty part of the other complex point, i.e.,

$$\begin{array}{ll} (\forall C \in \{A^\circ, A^-\}: & C \cap B^\circ \neq \varnothing \ \vee \ C \cap B^- \neq \varnothing) \ \wedge \\ (\forall D \in \{B^\circ, B^-\}: & A^\circ \cap D \neq \varnothing \ \vee \ A^- \cap D \neq \varnothing) \end{array}$$

Proof. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup B^- = \mathbb{R}^2$. That is, the complex point A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Since only non-empty object parts are considered, the interior and the exterior of A, respectively B, must hence intersect at least either the interior or the exterior or both parts of B, respectively A. □

Lemma 5.1.1 means that the second row and the second column of an intersection matrix only yield empty intersections so that we do not have to consider them any further. Lemma 5.1.2 indicates that the exteriors of two point objects always intersect. This leads to a value 1 at position $(3,3)$ in each valid matrix. Lemma 5.1.3 says that in the first and third row and in the first and third column of a matrix at least one "corner" intersection must yield true so that we find the value 1 in the matrix there.

The application of the three constraint rules results in a reduction of the 512 possible intersection matrices to five remaining matrices, which describe the characteristic features of the existing topological relationships between complex points. The corresponding matrices and their geometric interpretations are given in Table 4.

Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

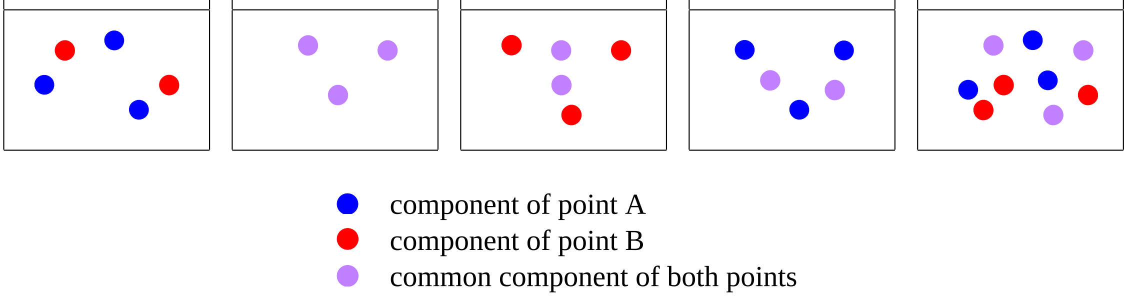


Table 4: The five topological relationships between two complex points.

Due to the small number of topological relationships, with each matrix we can associate a name for the corresponding topological predicate. Matrix 1 describes the relationship *disjoint*, matrix 2 the relationship *equal*, matrix 3 the relationship *inside*, matrix 4 the relationship *contains*, and matrix 5 the relationship *overlap*. The two topological relationships *disjoint* and *equal* between two *simple* point objects are included in matrix 1 and matrix 2, respectively.

Finally, we can summarize our result as follows:

Theorem 5.1 Based on the 9-intersection model, five different topological relationships can be identified between two complex *point* objects.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.1.1, 5.1.2, and 5.1.3, reduce the number of the 512 possible intersection matrices to five matrices. The ability to draw prototypes of the corresponding five topological configurations (Table 4) proves that the constraint rules are complete. □

5.2 Topological Relationships between Two Complex Lines

Next, we discuss topological relationships between two non-empty, complex lines A and B defined according to Section 3.2. We pursue the same strategy as before and present constraint rules to filter out non-existent topological configurations.

Lemma 5.2.1 The exteriors of two complex line objects always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. We know that $\overline{A} \cup A^- = A \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = B \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $A = \mathbb{R}^2$, or (ii) $B = \mathbb{R}^2$, or (iii) $A \cup B = \mathbb{R}^2$. These situations are all impossible, since A, B, and $A \cup B$ as bounded, one-dimensional shapes of finite length are unable to cover the unbounded, two-dimensional plane $\mathbb{R}^2$. □

Lemma 5.2.2 The interior of a complex line object intersects either the interior, the boundary, or the exterior of the other line object, i.e.,

$$(A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing) \ \wedge$$
$$(A^\circ \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap B^\circ \neq \varnothing)$$

Proof. Intuitively, this constraint means that neither the first row nor the first column of a 3×3-intersection matrix may only contain zeros. Assuming that the constraint rule is false. Then $(A^\circ \cap B^\circ = \varnothing \ \wedge \ A^\circ \cap \partial B = \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing) \ \vee \ (A^\circ \cap B^\circ = \varnothing \ \wedge \ \partial A \cap B^\circ = \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing)$. We show that the first argument of the disjunction (similar for the second argument) leads to a contradiction. The first argument can be summarized to $A^\circ \cap (B^\circ \cup \partial B \cup B^-) = A^\circ \cap \mathbb{R}^2 = \varnothing$. This is a contradiction to the assumed non-emptiness of a *line* object requiring that $A^\circ \neq \varnothing$ and $A^\circ \cap \mathbb{R}^2 = A^\circ$, i.e., $A^\circ \subset \mathbb{R}^2$. □

Lemma 5.2.3 If the boundary of a complex line object intersects the interior of another line object, also its exterior intersects the interior of the other line object, i.e.,

$$((\partial A \cap B^\circ \neq \varnothing \Rightarrow A^- \cap B^\circ \neq \varnothing) \ \wedge \ (A^\circ \cap \partial B \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing))$$
$$\Leftrightarrow \quad ((\partial A \cap B^\circ = \varnothing \ \vee \ A^- \cap B^\circ \neq \varnothing) \ \wedge \ (A^\circ \cap \partial B = \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing))$$

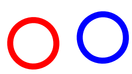
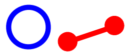
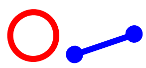
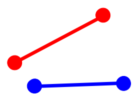
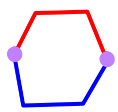
Proof. Without loss of generality, let P be an endpoint of the boundary of A located in the interior of B. From P exactly one curve of A starts or ends. Either P divides a curve of B into two subcurves, or P is endpoint of more than one curve of B. Hence, in P at least two curves of B end. Since the curve of A can coincide with at most one of the curves of B, at least one of the curves of B must be situated in the exterior of A. □

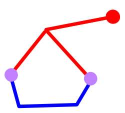
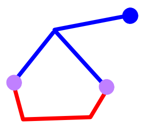
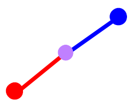
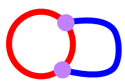
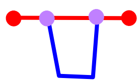
Lemma 5.2.4 If the boundary of a complex line object intersects the exterior of another line object, also its interior intersects the exterior of the other line object, i.e.,

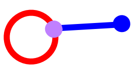
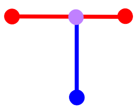
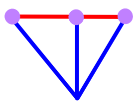
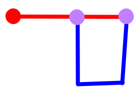
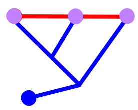
$$
\begin{aligned}
&((\partial A \cap B^- \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing) \wedge (A^- \cap \partial B \neq \varnothing \Rightarrow A^- \cap B^\circ \neq \varnothing)) \\
\Leftrightarrow\ &((\partial A \cap B^- = \varnothing \vee A^\circ \cap B^- \neq \varnothing) \wedge (A^- \cap \partial B = \varnothing \vee A^- \cap B^\circ \neq \varnothing))
\end{aligned}
$$

Proof. For each point $p \in B^-$ we can find a neighborhood $N^*(p)$ such that $N^*(p) \subset B^-$. If $p \in \partial A$, in *each* neighborhood of p we must find points of A°, since a curve of A starts at p. Hence, interior points of A exist that intersect B^-. □

An evaluation of all 512 3×3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that 82 matrices satisfy these rules and thus represent the possible topological relationships between two complex lines. The matrices and their geometric interpretations are shown in Table 5. The 33 topological relationships between simple lines [16, 11, 5] are contained and correspond to the intersection matrices with the numbers 4, 5, 8, 10, 12, 14, 19, 20, 23, 25-29, 36, 40, 42, 48, 50, 54, 56, 58, 60, 66-68, 72, 73, and 75-79. If we consider line objects as connected components, as it is also done in [16], the additional 24 topological relationships are also contained and correspond to the intersection matrices with the numbers 6, 7, 13, 15, 16, 21, 22, 24, 30-32, 38, 51-53, 59, 61, 62, 70, 71, 74, and 80-82.

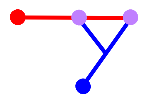
Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
				

Matrix 6	Matrix 7	Matrix 8	Matrix 9	Matrix 10
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$
				

Matrix 11	Matrix 12	Matrix 13	Matrix 14	Matrix 15
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
				

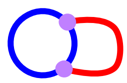
Matrix 16

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



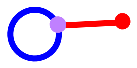
Matrix 17

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



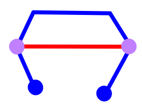
Matrix 18

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



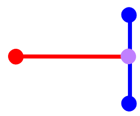
Matrix 19

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



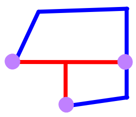
Matrix 20

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



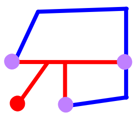
Matrix 21

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



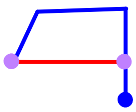
Matrix 22

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



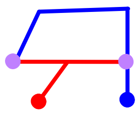
Matrix 23

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



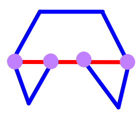
Matrix 24

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



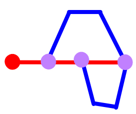
Matrix 25

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



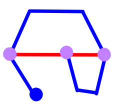
Matrix 26

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



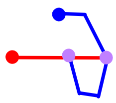
Matrix 27

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



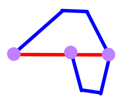
Matrix 28

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



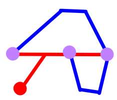
Matrix 29

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



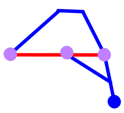
Matrix 30

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



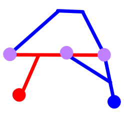
Matrix 31

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix 32

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

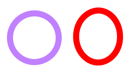


Matrix 33

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

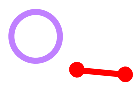
Matrix 34

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix 35

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

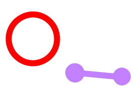


Matrix 36

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

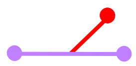
Matrix 37

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



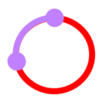
Matrix 38

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



Matrix 39

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

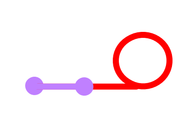


Matrix 40

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

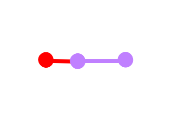
Matrix 41

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



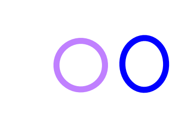
Matrix 42

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



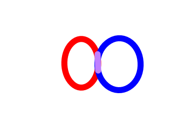
Matrix 43

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



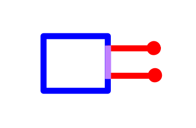
Matrix 44

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



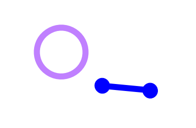
Matrix 45

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



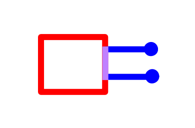
Matrix 46

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



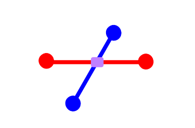
Matrix 47

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



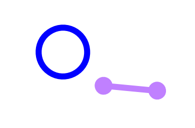
Matrix 48

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



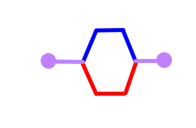
Matrix 49

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



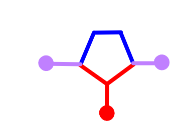
Matrix 50

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



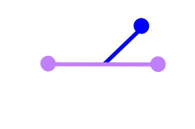
Matrix 51

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



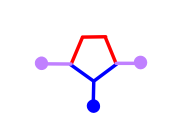
Matrix 52

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



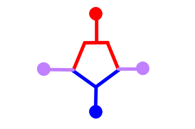
Matrix 53

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



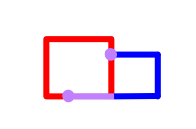
Matrix 54

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



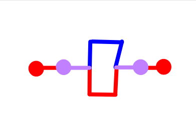
Matrix 55

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



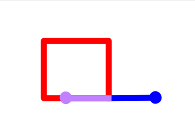
Matrix 56

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



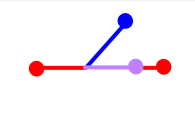
Matrix 57

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



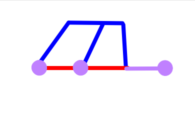
Matrix 58

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



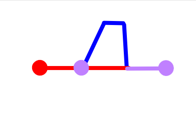
Matrix 59

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



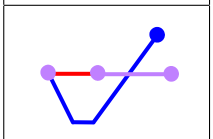
Matrix 60

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



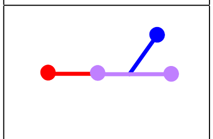
Matrix 61

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



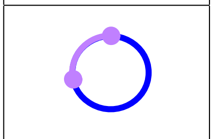
Matrix 62

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



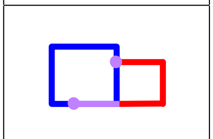
Matrix 63

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



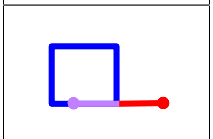
Matrix 64

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix 65

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



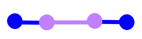
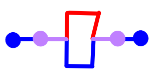
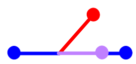
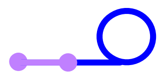
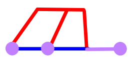
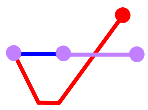
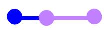
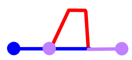
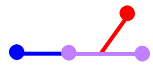
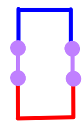
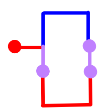
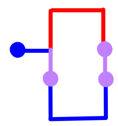
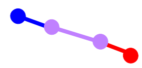
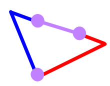
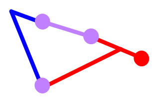
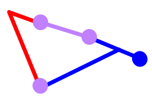
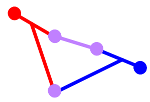
Matrix 66	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
Matrix 67	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 68	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 69	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
Matrix 70	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 71	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 72	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
Matrix 73	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 74	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 75	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 76	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 77	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 78	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 79	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 80	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 81	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 82	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	

Table 5: The 82 topological relationships between two complex lines.

Finally, we can summarize our result as follows:

Theorem 5.2 Based on the 9-intersection model, 82 different topological relationships can be identified between two complex *line* objects.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.2.1 to 5.2.4, reduce the number of the 512 possible intersection matrices to 82 matrices. The ability to draw prototypes of the corresponding 82 topological configurations (Table 5) proves that the constraint rules are complete. □

5.3 Topological Relationships between Two Complex Regions

In this section we identify those topological relationships that can be realized between two non-empty, complex regions A and B defined according to Section 3.3. We present constraint rules that exclude non-existent topological configurations. Note that a *part* of a complex region denotes either its boundary, interior, or exterior and that all parts are non-empty (see Section 3.3).

Lemma 5.3.1 Each part of a complex region intersects at least one part of the other complex region, i.e.,

$$(\forall C \in \{A^\circ, \partial A, A^-\} : C \cap B^\circ \neq \varnothing \ \vee \ C \cap \partial B \neq \varnothing \ \vee \ C \cap B^- \neq \varnothing) \ \wedge$$
$$(\forall D \in \{B^\circ, \partial B, B^-\} : A^\circ \cap D \neq \varnothing \ \vee \ \partial A \cap D \neq \varnothing \ \vee \ A^- \cap D \neq \varnothing)$$

Proof. We know that $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup \partial B \cup B^- = \mathbb{R}^2$. That is, the complex region A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Hence, each part of A, respectively B, must intersect at least one part of B, respectively A. □

Since a row in the matrix represents the possible intersections of a part of A with all parts of B and since a column represents the possible intersections of a part of B with all parts of A, in each row and in each column at least one intersection must yield true so that we find the value 1 in the matrix there.

In the following lemma we formulate a constraint rule on the basis of subsets. Since the 9-intersection model rests on the equality or inequality of the intersection of sets, we express the subset relationships in terms of the nine intersections and show the equivalence.

Lemma 5.3.2 Neither the interior nor the exterior of a complex region can be completely contained in or equal to the boundary of the other complex region, i.e.,

$$\begin{aligned} & A^\circ \not\subseteq \partial B \ \wedge \ A^- \not\subseteq \partial B \ \wedge \ B^\circ \not\subseteq \partial A \ \wedge \ B^- \not\subseteq \partial A \\ \Leftrightarrow \quad & (A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing) \ \wedge \ (A^- \cap B^\circ \neq \varnothing \ \vee \ A^- \cap B^- \neq \varnothing) \ \wedge \\ & (A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^- \cap B^\circ \neq \varnothing) \ \wedge \ (A^\circ \cap B^- \neq \varnothing \ \vee \ A^- \cap B^- \neq \varnothing) \end{aligned}$$

Proof. This lemma follows from the fact that the dimension of a boundary with its linear structure is less than the dimensions of the interior and the exterior with their areal structures.

We show the equivalence considering the subexpression $A^\circ \not\subseteq \partial B \Leftrightarrow (A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing)$. For the other subexpressions the argumentation is similar. We first show the "$\Rightarrow$" direction. If $A^\circ \not\subseteq \partial B$, then $A^\circ \cap (\mathbb{R}^2 - \partial B) = A^\circ \cap (B^\circ \cup B^-) \neq \varnothing$. Due to distributivity $A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing$ follows. For the "$\Leftarrow$" direction we argue like before but in the opposite direction. □

Lemma 5.3.3 The exteriors of two complex region objects always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. Let $r_1, r_2 \in \mathbb{R}^+$ be numbers according to Definition 3.11 (iii) such that the norms of A and B, respectively, are less than r_1 and r_2, respectively. Let $r_m = \max(r_1, r_2)$. Since we can find a point $p = (x, y) \in \mathbb{R}^2$ with $\sqrt{x^2 + y^2} > r_m$, this point is in the exterior of A and in the exterior of B. Hence, we obtain that $p \in A^- \cap B^-$ which proves the lemma. □

Lemma 5.3.4 The boundaries of two complex regions are equal if, and only if, the interiors and the exteriors, respectively, of both regions are equal, i.e.,

$$
\begin{array}{ll}
& (\partial A = \partial B \Leftrightarrow A^\circ = B^\circ \ \wedge \ A^- = B^-) \\
\Leftrightarrow & (c \Leftrightarrow d) \Leftrightarrow ((c \ \wedge \ d) \ \vee \ (\neg c \ \wedge \ \neg d)) \text{ where} \\
& c = A^\circ \cap \partial B = \varnothing \ \wedge \ \partial A \cap B^\circ = \varnothing \ \wedge \ \partial A \cap \partial B \neq \varnothing \ \wedge \\
& \quad \partial A \cap B^- = \varnothing \ \wedge \ A^- \cap \partial B = \varnothing \text{ and} \\
& d = A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap \partial B = \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \ \wedge \\
& \quad \partial A \cap B^\circ = \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing \ \wedge \ \partial A \cap B^- = \varnothing \ \wedge \\
& \quad A^- \cap \partial B = \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing
\end{array}
$$

Proof. This very special constraint rule expresses that complex regions are uniquely characterized by their boundaries. This is ensured by the Jordan Curve Theorem [23]. □

Lemma 5.3.5 If the boundary of a complex region intersects the interior of the other complex region, both its interior and its exterior intersect the interior of the other region, i.e.,

$$
\begin{array}{ll}
& ((\partial A \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing)) \ \wedge \\
& \ (A^\circ \cap \partial B \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing))) \\
\Leftrightarrow & ((\partial A \cap B^\circ = \varnothing \ \vee \ (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing)) \ \wedge \\
& \ (A^\circ \cap \partial B = \varnothing \ \vee \ (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing)))
\end{array}
$$

Proof. Let $p \in \partial A \cap B^\circ$. Since $p \in \partial A$, according to the boundary definition in Definition 3.9, for each neighborhood $N^*(p)$ holds that $N^*(p) \cap A^\circ \neq \varnothing$ and $N^*(p) \cap A^- \neq \varnothing$. Since $p \in B^\circ$, there is a neighborhood $M^*(p)$ which is fully contained in B°. The non-empty intersection of one of the $N^*(p)$ and $M^*(p)$ proves the lemma. □

Lemma 5.3.6 If the boundary of a complex region intersects the exterior of the other complex region, both its interior and its exterior intersect the exterior of the other region, i.e.,

$$
\begin{array}{ll}
& ((\partial A \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \ \wedge \\
& \ (A^- \cap \partial B \neq \varnothing \Rightarrow (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing))) \\
\Leftrightarrow & ((\partial A \cap B^- = \varnothing \ \vee \ (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \ \wedge \\
& \ (A^- \cap \partial B = \varnothing \ \vee \ (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)))
\end{array}
$$

Proof. The argumentation is similar as for the previous constraint. But we can also argue by employing the Jordan Curve Theorem [23]. Due to this theorem, on each side of the boundary of a region there is either the region's interior or exterior. On both sides of a line intersecting the exterior of this region, we find the exterior of the region. If the line is part of the boundary of another region, we obtain the intersection of both regions' exteriors and the intersection between the interior of the first region and the exterior of the other region. □

Lemma 5.3.7 Either the boundaries of two complex regions intersect, or the boundary of one region intersects the exterior of the other region, i.e.,

$$
\partial A \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing
$$

Proof. Assuming that the constraint rule is false. Then $\partial A \cap \partial B = \varnothing \ \wedge \ \partial A \cap B^- = \varnothing \ \wedge \ A^- \cap \partial B = \varnothing$. With Lemma 5.3.1, $\partial A \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap \partial B \neq \varnothing$ holds. Without loss of generality, let us consider a point $p \in A^\circ \cap \partial B$ and an infinite ray s emanating from p in an arbitrary direction. Since the component (face) of A containing p is bounded, s encounters the boundary of A in a point, say, q. This boundary could potentially intersect the exterior, the boundary, or the interior of B. But according to our assumption, the first two cases cannot hold so that q must lie inside the interior of B. We obtain a similar situation as before, except for the fact that now A and B change their roles. We continue to observe the course of s: the ray over and over again alternately encounters a point of $A^\circ \cap \partial B$ and then a point of $\partial A \cap B^\circ$. Since the ray can be prolonged arbitrarily, A and B must be unbounded or consist of infinitely many components. But this is a contradiction to the definition of the *region* data type. □

Lemma 5.3.8 If the interiors of two complex regions intersect, the interior of one region also intersects the boundary of the other region, or the regions' boundaries intersect, i.e.,

$$\begin{array}{ll} & (A^\circ \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \\ \Leftrightarrow & (A^\circ \cap B^\circ = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing) \end{array}$$

Proof. Without loss of generality, let us consider a component of the first region and a component of the second region with intersecting interiors. We have to distinguish three situations. First, if the interiors of both components are equal, also their boundaries are equal and hence intersect. Consequently, also the regions' boundaries intersect. Second, if the interiors of both components but not their boundaries intersect, one component is contained in the other. Since this is a proper containment (otherwise the boundaries would intersect), the boundary of one component must be inside the interior of the other component. Consequently, the interior of one region intersects the boundary of the other region. Third, if the interiors as well as the boundaries of the two components intersect, the remaining two conclusions of the constraint rule hold. □

Lemma 5.3.9 If the interior of a complex region intersects the exterior of the other region, either the interior of the first region intersects the boundary of the second region, or the boundary of the first region intersects the exterior of the second region, or both regions' boundaries intersect, i.e.,

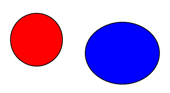
$$\begin{array}{ll} & ((A^\circ \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \ \wedge \\ & (A^- \cap B^\circ \neq \varnothing \Rightarrow (\partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing))) \\ \Leftrightarrow & ((A^\circ \cap B^- = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing) \ \wedge \\ & (A^- \cap B^\circ = \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \end{array}$$

Proof. If there is an intersection between the interior of a complex region and the exterior of the other complex region, a few different situations for each component causing the intersection can be distinguished. The first situation is that a component partially intersects the interior and the exterior of the other region. Then the boundary of the other region intersects the interior of the first region. The second situation is that the interior of a component lies completely inside the exterior of the other region. Several cases can now be distinguished. The first case is that also the boundary (and thus the entire component) lies inside and consequently intersects the exterior of the other region. The second case is that the boundary of a component lies only partially inside the exterior of the other region. Again we obtain an intersection between boundary and exterior. The third case is that the boundary of a component intersects the boundary of the other region. Note that the boundary of the component cannot cross the interior of the other region, since then the interior of the component would not be entirely within the exterior of the other region. □

An evaluation of all 512 3×3-intersection matrices against these nine constraint rules with the aid of a simple test program reveals that 33 matrices satisfy these rules and thus represent the possible topological relationships between two complex regions. The matrices and their geometric interpretations are shown in Table 6. The eight topological relationships between simple regions [13, 10, 14, 15] are contained and correspond to the intersection matrices with the numbers 1, 4, 5, 7, 9, 19, 24, and 33.

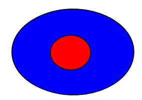
Matrix 1

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



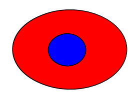
Matrix 2

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



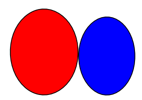
Matrix 3

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



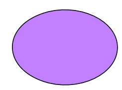
Matrix 4

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



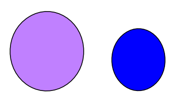
Matrix 5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



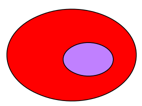
Matrix 6

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



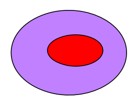
Matrix 7

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



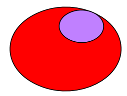
Matrix 8

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



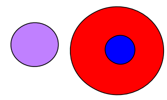
Matrix 9

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



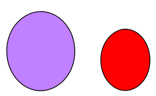
Matrix 10

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



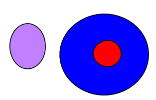
Matrix 11

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



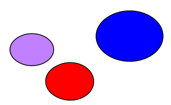
Matrix 12

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



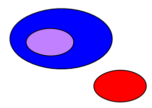
Matrix 13

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



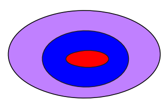
Matrix 14

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



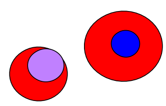
Matrix 15

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



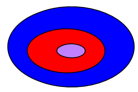
Matrix 16

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



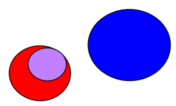
Matrix 17

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



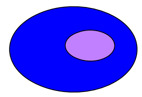
Matrix 18

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



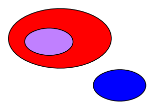
Matrix 19

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



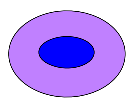
Matrix 20

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



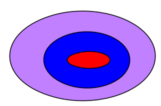
Matrix 21

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



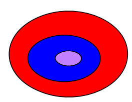
Matrix 22

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



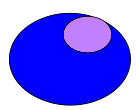
Matrix 23

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



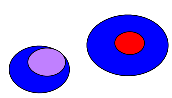
Matrix 24

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



Matrix 25

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



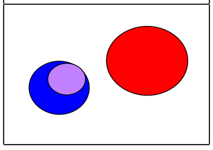
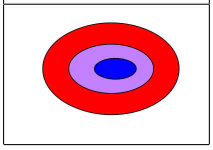
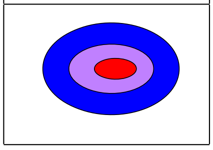
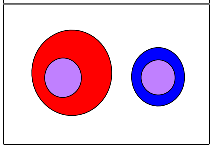
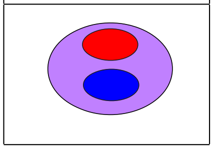
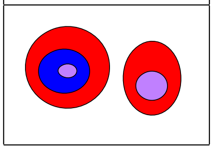
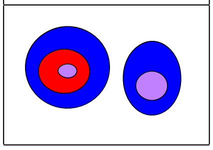
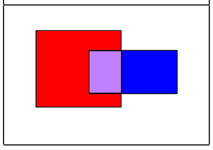
Matrix	Intersection matrix	Prototype
Matrix 26	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 27	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 28	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 29	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 30	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 31	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
Matrix 32	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	
Matrix 33	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	

Table 6: The 33 topological relationships between two complex regions.

Finally, we can summarize our result as follows:

Theorem 5.3 Based on the 9-intersection model, 33 different topological relationships can be identified between two complex *region* objects.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.3.1 to 5.3.9, reduce the number of the 512 possible intersection matrices to 33 matrices. The ability to draw prototypes of the corresponding 33 topological configurations (Table 6) proves that the constraint rules are complete. □

6 Topological Relationships between Two Complex Spatial Objects of Distinct Type

Next we analyze the topological relationships between two non-empty complex spatial objects that have *different* types and thus different properties and dimensions. This leads to the six type combinations *point*/*line* (Section 6.1), *point*/*region* (Section 6.2), *line*/*region* (Section 6.3), *line*/*point*, *region*/*point*, and *region*/*line*. The latter three type combinations are symmetric counterparts of the first three type combinations so that they do not have to be treated separately. It is obvious that a number of topological relationships cannot exist. For example, higher-dimensional objects can never be located inside lower-dimensional objects.

6.1 Topological Relationships between a Complex Point and a Complex Line

Constraint rules for points and lines can also be based on the 9-intersection matrix. In the following, we assume that A is a non-empty object of type *point* and B is a non-empty object of type *line*.

Lemma 6.1.1 All intersections comprising an operand with the boundary operator of the complex point object yield the empty set, i.e.,

$$\forall D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing$$

Proof. According to the definition of a complex point $\partial A = \varnothing$ holds. Hence, the intersection of the empty set with any component of B yields the empty set. □

Lemma 6.1.2 The intersection of the interior of the complex line object and the exterior of the complex point object cannot be empty, i.e.,

$$A^- \cap B^\circ \neq \varnothing$$

Proof. Assuming that the constraint rule is wrong. Then $A^- \cap B^\circ = \varnothing$. Since we know that $\partial A = \varnothing$, we can conclude that $A^\circ = B^\circ$. This leads to a contradiction since the finite set representing the point object $A = A^\circ$ cannot cover the infinite set representing B°. □

Lemma 6.1.3 The exteriors of the complex point and the complex line always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. We know that $\overline{A} \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $\overline{A} = \mathbb{R}^2$, or (ii) $\overline{B} = \mathbb{R}^2$, or (iii) $\overline{A} \cup \overline{B} = \mathbb{R}^2$. The situations are all impossible, since A, B, and hence $A \cup B$ are bounded, but $\mathbb{R}^2$ is unbounded. □

Lemma 6.1.4 The interior of the complex point intersects at least one part of the complex line, i.e.,

$$A^\circ \cap \partial B \neq \varnothing \;\vee\; A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing$$

Proof. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $\partial B \cup B^\circ \cup B^- = \mathbb{R}^2$. Since only non-empty object parts of both objects are considered, we obtain $A^\circ \cap \mathbb{R}^2 = A^\circ \cap (\partial B \cup B^\circ \cup B^-) \neq \varnothing$. This statement is equivalent to the constraint rule. □

An evaluation of all 512 3×3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that 14 matrices satisfy these rules and thus represent the possible topological relationships between a complex point and a complex line. The matrices and their geometric interpretations are shown in Table 7. Between a *simple* point and a *simple* line we can distinguish three topological relationships. Either a simple point and a simple line are disjoint, or the simple point is located in one of the endpoints of the simple line, or the simple point is situated in the interior of the simple line. These topological predicates are contained in the 14 general ones and correspond to the matrices 2, 4, and 8, respectively.

Finally, we can summarize our result as follows:

Theorem 6.1 Based on the 9-intersection model, 14 different topological relationships can be identified between a complex *point* object and a complex *line* object.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.1.1 to 6.1.4, reduce the number of the 512 possible intersection matrices to 14 matrices. The ability to draw prototypes of the corresponding 14 topological configurations (Table 7) proves that the constraint rules are complete. □

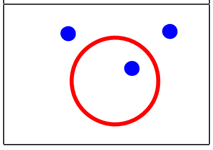
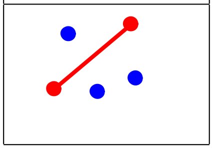
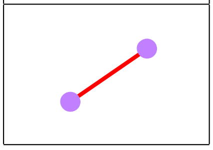
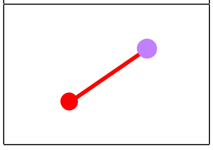
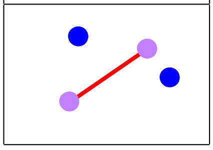
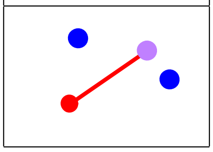
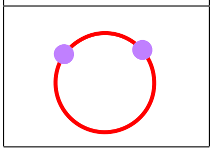
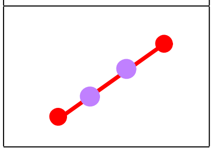
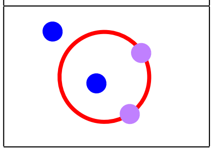
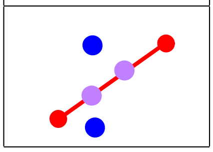
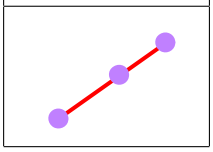
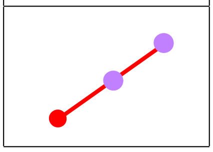
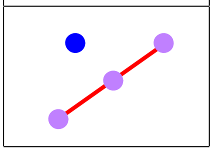
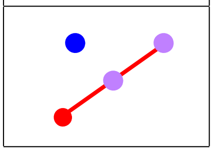
Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
				
Matrix 6	**Matrix 7**	**Matrix 8**	**Matrix 9**	**Matrix 10**
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$
				
Matrix 11	**Matrix 12**	**Matrix 13**	**Matrix 14**	
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
				

Table 7: The 14 topological relationships between a complex point and a complex line.

6.2 Topological Relationships between a Complex Point and a Complex Region

The point-in-polygon test is probably the most classical representative of a topological predicate between a point and a polygon. In the following, we assume that A is a non-empty object of type *point* and B is a non-empty object of type *region*. The following constraint rules lead us to all topological predicates between a complex point object and a complex region object.

Lemma 6.2.1 The exteriors of the complex point and the complex region always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. The argumentation is the same as for Lemma 6.1.3. □

Lemma 6.2.2 All intersections comprising an operand with the boundary operator of the complex point object yield the empty set, i.e.,

$$\forall D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing$$

Proof. The argumentation is the same as for Lemma 6.1.1. □

Lemma 6.2.3 The interior and the boundary of the complex region object intersect the exterior of the point object, i.e.,

$$A^- \cap B^\circ \neq \varnothing \;\wedge\; A^- \cap \partial B \neq \varnothing$$

Proof. Both the interior and the boundary of a complex region object are infinite point sets whereas the complex point object is a finite point set. Hence, the complex point object can cover neither the interior nor the boundary of the complex region object so that its exterior must intersect these region parts. □

Lemma 6.2.4 The interior of the complex point intersects at least one part of the complex region, i.e.,

$$A^\circ \cap \partial B \neq \varnothing \;\vee\; A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing$$

Proof. The argumentation is the same as for Lemma 6.1.4. □

An evaluation of all 512 3×3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that 7 matrices satisfy these rules and thus represent the possible topological relationships between a complex point and a complex region. The matrices and their geometric interpretations are shown in Table 8. Between a *simple* point and a *simple* region we can distinguish three topological relationships. Either a simple point and a simple region are disjoint (we also say the point is *outside* the region), or the simple point is located *on* the boundary of the simple region, or the simple point is *inside* the simple region. These topological predicates are contained in the 7 general ones and correspond to the matrices 1, 2, and 4, respectively.

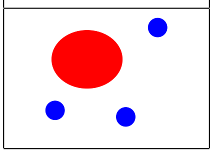
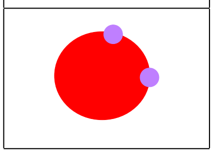
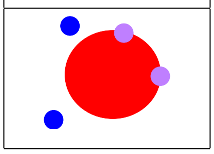
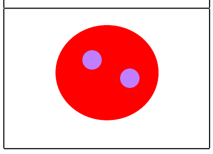
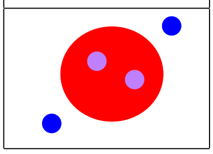
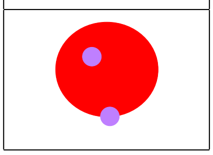
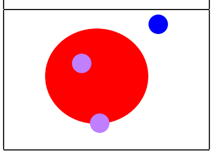
Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$
				
Matrix 6	**Matrix 7**			
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$			
				

Table 8: The 7 topological relationships between a complex point and a complex region.

Finally, we can summarize our result as follows:

Theorem 6.2 Based on the 9-intersection model, 7 different topological relationships can be identified between a complex *point* object and a complex *region* object.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.2.1 to 6.2.4, reduce the number of the 512 possible intersection matrices to 7 matrices. The ability to draw prototypes of the corresponding 7 topological configurations (Table 8) proves that the constraint rules are complete. □

6.3 Topological Relationships between a Complex Line and a Complex Region

In the following last case, we assume that A is a non-empty object of type *line* and B is a non-empty object of type *region*. The following constraint rules identify all topological predicates between a complex line object and a complex region object.

Lemma 6.3.1 The exteriors of the complex line and the complex region always intersect with each other, i.e.,

$$A^- \cap B^- \neq \varnothing$$

Proof. The argumentation is the same as for Lemma 6.1.3. □

Lemma 6.3.2 The interior of the complex region always intersects the exterior of the complex line, i.e.,

$$A^- \cap B^\circ \neq \varnothing$$

Proof. Assuming that this constraint rule is wrong. Then $A^- \cap B^\circ = \varnothing$, and we can conclude that $A \supseteq B^\circ$. From this we obtain that $\forall\, p \in B^\circ\ \exists \varepsilon \in \mathbb{R}^+ : N_\varepsilon(p) \subseteq B^\circ \Rightarrow N_\varepsilon(p) \subseteq A$. This leads to a contradiction since $\forall\, p \in B^\circ\ \forall \varepsilon \in \mathbb{R}^+ : N_\varepsilon(p) \not\subseteq A$. □

Intuitively, a line object as a one-dimensional, linear entity cannot cover a region object, which is a two-dimensional, areal entity.

Lemma 6.3.3 The interior or the exterior of the complex line intersects the boundary of the complex region, i.e.,

$$A^\circ \cap \partial B \neq \varnothing\ \vee\ A^- \cap \partial B \neq \varnothing$$

Proof. We know that $\partial B \neq \varnothing$ and that hence $\mathbb{R}^2 \cap \partial B \neq \varnothing$. Since $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$, we obtain that $(A^\circ \cup \partial A \cup A^-) \cap \partial B \neq \varnothing$. This leads to $A^\circ \cap \partial B \neq \varnothing\ \vee\ \partial A \cap \partial B \neq \varnothing\ \vee\ A^- \cap \partial B \neq \varnothing$. Since ∂A is a finite point set and ∂B is an infinite point set, either $\partial A \subset \partial B$ or $\partial A \cap \partial B = \varnothing$. This means that the constraint rule $A^\circ \cap \partial B \neq \varnothing\ \vee\ A^- \cap \partial B \neq \varnothing$ must hold. □

Lemma 6.3.4 The interior of the complex line intersects at least one part of the complex region, i.e.,

$$A^\circ \cap \partial B \neq \varnothing\ \vee\ A^\circ \cap B^\circ \neq \varnothing\ \vee\ A^\circ \cap B^- \neq \varnothing$$

Proof. The argumentation is the same as for Lemma 6.1.4. □

Lemma 6.3.5 If the boundary of the complex line intersects the interior of the complex region, also its interior intersects the interior of the complex region, i.e.,

$$\begin{aligned} & (\partial A \cap B^\circ \neq \varnothing \Rightarrow A^\circ \cap B^\circ \neq \varnothing) \\ \Leftrightarrow\ & (\partial A \cap B^\circ = \varnothing \ \vee\ A^\circ \cap B^\circ \neq \varnothing) \end{aligned}$$

Proof. Without loss of generality, let $p \in \partial A \cap B^\circ$. Since $p \in B^\circ$, an $\varepsilon \in \mathbb{R}^+$ exists such that $N_\varepsilon(p) \subset B^\circ$. Fixing such an ε, and because a continuous curve with an infinite number of points starts in p, we obtain that $N_\varepsilon(p) \cap A^\circ \neq \varnothing$. This leads to the conclusion that $A^\circ \cap B^\circ \neq \varnothing$. □

Lemma 6.3.6 If the boundary of the complex line intersects the exterior of the complex region, also its interior intersects the exterior of the complex region, i.e.,

$$\begin{aligned} & (\partial A \cap B^- \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing) \\ \Leftrightarrow\ & (\partial A \cap B^- = \varnothing \ \vee\ A^\circ \cap B^- \neq \varnothing) \end{aligned}$$

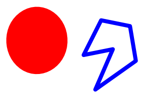
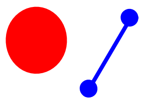
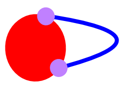
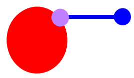
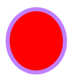
Proof. The argumentation is analogous to the argumentation for the constraint rule in Lemma 6.3.5. □

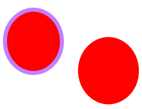
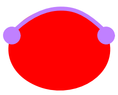
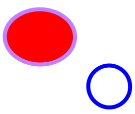
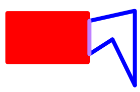
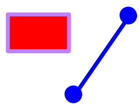
Lemma 6.3.7 If the boundary of the complex line intersects the boundary of the complex region, also its exterior intersects the boundary of the complex region, i.e.,

$$\begin{aligned} & (\partial A \cap \partial B \neq \varnothing \Rightarrow A^- \cap \partial B \neq \varnothing) \\ \Leftrightarrow\ & (\partial A \cap \partial B = \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing) \end{aligned}$$

Proof. The boundary of a region is a line object whose components are all closed curves. Hence, this line object only consists of interior points. This leads to the case we have discussed in Lemma 5.2.3. □

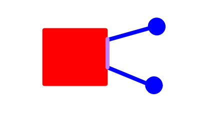
An evaluation of all 512 3×3-intersection matrices against these seven constraint rules with the aid of a simple test program reveals that 43 matrices satisfy these rules and thus represent the possible topological relationships between a complex line and a complex region. The matrices and their geometric interpretations are shown in Table 11. Between a *simple* line and a *simple* region we can distinguish 19 topological relationships [16]. These topological predicates are contained in the 43 general ones and correspond to the matrices 2-4, 7, 11-13, 15-17, 28, 30, 31, 35-37, 39, 41, and 42, respectively.

Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
				

Matrix 6	Matrix 7	Matrix 8	Matrix 9	Matrix 10
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
				

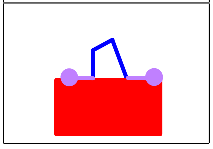
Matrix 11

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



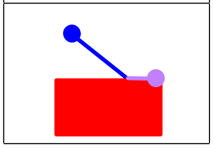
Matrix 12

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



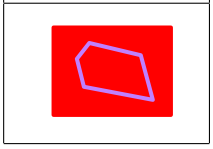
Matrix 13

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



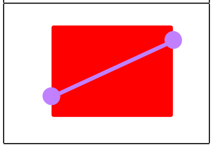
Matrix 14

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



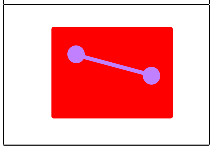
Matrix 15

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



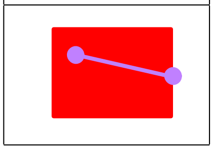
Matrix 16

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



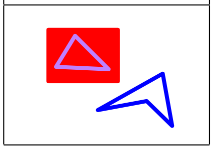
Matrix 17

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



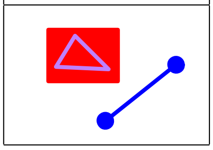
Matrix 18

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



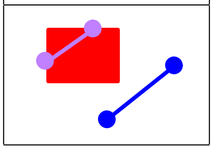
Matrix 19

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



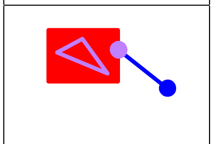
Matrix 20

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



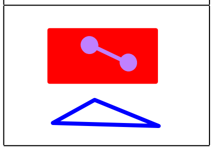
Matrix 21

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



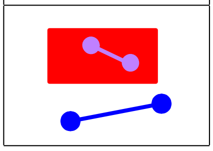
Matrix 22

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



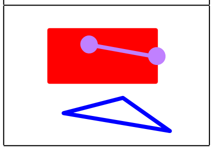
Matrix 23

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



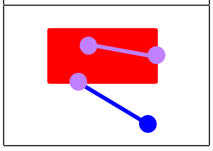
Matrix 24

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



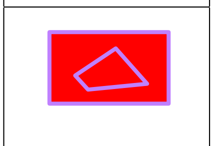
Matrix 25

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



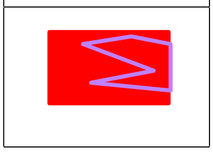
Matrix 26

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



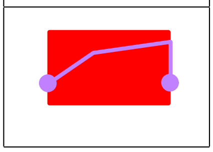
Matrix 27

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



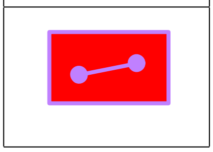
Matrix 28

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



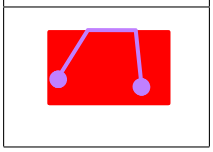
Matrix 29

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



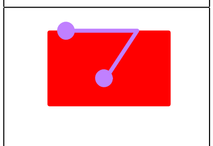
Matrix 30

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



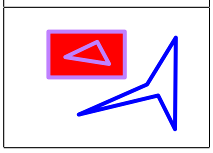
Matrix 31

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



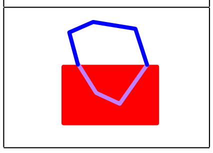
Matrix 32

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



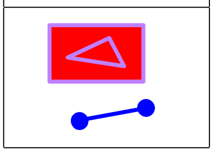
Matrix 33

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



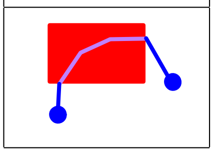
Matrix 34

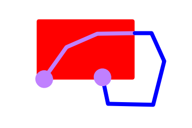
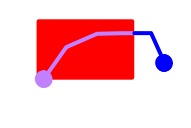
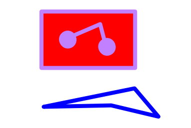
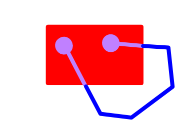
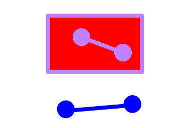
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix 35

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



Matrix 36	Matrix 37	Matrix 38	Matrix 39	Matrix 40
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
				

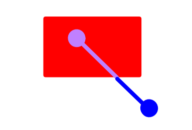
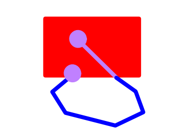
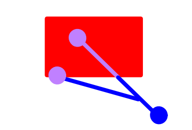
Matrix 41	Matrix 42	Matrix 43
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
		

Table 11: The 43 topological relationships between a complex line and a complex region.

Finally, we can summarize our result as follows:

Theorem 6.3 Based on the 9-intersection model, 43 different topological relationships can be identified between a complex *line* object and a complex *region* object.

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.3.1 to 6.3.7, reduce the number of the 512 possible intersection matrices to 43 matrices. The ability to draw prototypes of the corresponding 43 topological configurations (Table 11) proves that the constraint rules are complete. □

7 Clustering of Topological Predicates

Based on the 9-intersection model and the predicate derivation mechanism described in Section 4, in Sections 5 and 6, we have systematically identified the topological relationships between any two spatial objects of the data types *point*, *line*, and *region* defined in Section 3. Objects of these data types have a much more complex internal structure than simple objects (Section 2.1). We have seen that topological predicates operating on complex spatial objects comprise, generalize, and extend the predicates found so far for simple object structures (see Section 2.2 and Table 3). Table 12 summarizes the number of predicates obtained for each type combination.

In the following, Section 7.1 addresses the problem of the large numbers and the manageability of the predicates in Table 12. The next two subsections present two possible solutions based on clustering rules (Section 7.2) and topological cluster predicates (Section 7.3) to cope with this quantity problem.

	complex point	complex line	complex region
complex point	5	14	7
complex line	14	82	43
complex region	7	43	33

Table 12: Number of topological predicates between two complex spatial objects.

7.1 The Quantity Problem

Compared to Table 3, unsurprisingly the number of topological predicates has increased for each combination of complex data types. Already for topological predicates on simple spatial objects, the large numbers of predicates have been considered a problem since they make it difficult for users to distinguish, remember, and handle them [7]. Consequently, this is, in particular, the case for the topological predicates on complex spatial objects. For example, whereas the 5 relationships between complex points are manageable and distinguishable by the user, this is certainly not the case for the 82 relationships between complex lines.

Frequently the user will not be interested in such a large, overwhelming collection of detailed predicates for a particular type combination and prefer a reduced and manageable set instead. We call such a reduced set of predicates *cluster*. Such a cluster should be user-defined and/or application-specific. It should be user-defined and thus flexible since the user should be able to select which predicates she wants to merge to more general predicates for her purposes. It should be application-specific since different applications may have a different understanding and thus definition of topological predicates carrying the same name. For example, one application could employ the *inside* predicate according to its original meaning, whereas another application could perhaps wish not to distinguish between *inside* and *coveredBy* but merge them and call the result predicate *inside* too.

Two possible solutions consist in (i) the design of *clustering rules* for topological predicates and (ii) in the explicit construction of user-defined and/or application-specific *topological cluster predicates* and *topological predicate clusters*. We will see that the first solution is more a designer approach than a user approach like the second solution and that its outcome implicitly leads to cluster predicates.

7.2 Implicit Definition of Topological Cluster Predicates through Clustering Rules

Clustering rules are relaxed constraint rules which do not take into account all nine intersections of the 9-intersection matrix. Only a predicate designer but not a user can usually specify them. They must be defined in a way so that they are mutually exclusive and cover all basic topological predicates identified in Sections 5 or 6 for the respective type combination. As an example, we will now give a specification of eight constraint rules that is generic in the sense that it is valid for all type combinations considered. Let $A \in \alpha$, $A \neq \varnothing$, and $B \in \beta$, $B \neq \varnothing$, for $\alpha, \beta \in \{point, line, region\}$. In the following, the notation p_c indicates that a predicate p is clustered. We define:

Clustering Rule 1 Two spatial objects are *disjoint*, if the parts of one object intersect at most with the exterior of the other object, i.e.,

$$disjoint_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ = \varnothing \,\wedge\, A^\circ \cap \partial B = \varnothing \,\wedge\, \partial A \cap B^\circ = \varnothing \,\wedge\, \partial A \cap \partial B = \varnothing$$

Clustering Rule 2 Two spatial objects *meet*, if both interiors do not intersect, but the interior or the boundary of one object intersects the boundary of the other object, i.e.,

$$meet_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ = \varnothing \;\wedge\; (A^\circ \cap \partial B \neq \varnothing \;\vee\; \partial A \cap B^\circ \neq \varnothing \;\vee\; \partial A \cap \partial B \neq \varnothing)$$

Clustering Rule 3 A spatial object is located *inside* another object, if (i) their interiors intersect, (ii) the inner object does not share anything with the exterior of the other object, (iii) the interior of the containing object is partially located in the exterior of the inner object, and (iv) the boundaries of both objects do not intersect, i.e.,

$$inside_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- = \varnothing \;\wedge\; A^- \cap B^\circ \neq \varnothing \;\wedge\; \partial A \cap \partial B = \varnothing$$

Clustering Rule 4 The relationships that are symmetric to $inside_c$ describe the predicate $contains_c$, i.e.,

$$contains_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- \neq \varnothing \;\wedge\; A^- \cap B^\circ = \varnothing \;\wedge\; \partial A \cap \partial B = \varnothing$$

Clustering Rule 5 A spatial object is *covered by* another object, if (i) their interiors intersect, (ii) the inner object does not share anything with the exterior of the other object, (iii) the interior of the containing object is partially located in the exterior of the inner object, and (iv) the boundaries of both objects intersect, i.e.,

$$coveredBy_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- = \varnothing \;\wedge\; A^- \cap B^\circ \neq \varnothing \;\wedge\; \partial A \cap \partial B \neq \varnothing$$

Clustering Rule 6 The relationships that are symmetric to $coveredBy_c$ describe the predicate $covers_c$, i.e.,

$$covers_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- \neq \varnothing \;\wedge\; A^- \cap B^\circ = \varnothing \;\wedge\; \partial A \cap \partial B \neq \varnothing$$

Clustering Rule 7 Two spatial objects are *equal*, if at most corresponding parts intersect, i.e.,

$$\begin{aligned} equal_c(A,B) \stackrel{\text{def}}{=} {} & A^\circ \cap \partial B = \varnothing \;\wedge\; A^\circ \cap B^- = \varnothing \;\wedge\; \partial A \cap B^\circ = \varnothing \;\wedge \\ & \partial A \cap B^- = \varnothing \;\wedge\; A^- \cap B^\circ = \varnothing \;\wedge\; A^- \cap \partial B = \varnothing \end{aligned}$$

Clustering Rule 8 Two spatial objects *overlap*, if the interior of each object intersects both the interior and the exterior of the other object, i.e.,

$$overlap_c(A,B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- \neq \varnothing \;\wedge\; A^- \cap B^\circ \neq \varnothing$$

In order to prove the mutual exclusion of the obtained topological cluster predicates as well as the complete coverage of all basic topological predicates by them, we do *not* argue on the basis of our definitions of clustering rules. Instead, the strategy is to match each clustering rule against all basic topological predicates, which are then associated with possible cluster predicates. Table 13 shows the result.

For each type combination it is easy to verify that each basic topological predicate is assigned to at most one topological cluster predicate and that there is no basic topological predicate that is not assigned to a cluster predicate. Hence, each basic predicate is assigned to exactly one cluster predicate, and all basic predicates are covered.

Cluster Predicate	point/ point	line/ line	region/ region	point/ line	point/ region	line/ region
$disjoint_c$	1	1-4	1	1, 2	1	1, 2
$meet_c$	–	5-32	2-4	3-6	2, 3	3-13
$covers_c$	–	49, 52, 69, 72	10, 20	–	–	–
$coveredBy_c$	–	37, 38, 41, 42	6, 8	–	–	15, 17, 28, 31
$inside_c$	3	34, 35, 39, 40	7	7, 8, 11, 12	4, 6	14, 16, 26, 27, 29, 30
$contains_c$	4	43, 46, 63, 66	17	–	–	–
$overlap_c$	5	44, 45, 47, 48, 50, 51, 53-62, 64, 65, 67, 68, 70, 71, 73-82	9, 11-16, 18, 19, 21-33	9, 10, 13, 14	5, 7	18-25, 32-43
$equal_c$	2	33, 36	5	–	–	–

Table 13: Generic topological cluster predicates and assigned basic topological predicates depending on the type combination and indicated by their matrix numbers.

7.3 Explicit Definition of Topological Cluster Predicates through Disjunctions

The lists of matrix numbers in Table 13 express that each clustered predicate is equal to a disjunction of basic predicates. Hence, from a user perspective, a cluster predicate summarizes basic topological predicates under a common name. In a database context, a user should be able to use disjunctions of basic topological predicates for explicitly constructing cluster predicates.

A first problem is how to address basic topological predicates, since they are nameless because of their large amount. For this purpose, we introduce the six type-combination specific predicates *tp_pp*, *tp_pl*, *tp_pr*, *tp_ll*, *tp_lr*, and *tp_rr* (the prefix "tp_" stands for "topological predicate", the last two letters denote the type combination) and parameterize them by their matrix number. Since we do not have predicate names and since the user can identify topological relationships only by looking at the example scenarios, we regard this as a practicable solution. For example, the two basic predicates *tp_pr*(5) and *tp_pr*(7) denote the two possible overlap situations between a point object and a region object.

A second problem is how user-defined topological cluster predicates can be specified in a database context. Here we propose an extension of the data description language (DDL) of SQL. For example, a DDL command

```
create tpclpred lr_inside(tp_lr(14), tp_lr(16), tp_lr(26),
                          tp_lr(27), tp_lr(29), tp_lr(30));
```

could specify a cluster predicate *lr_inside* which for two objects $A \in line$ and $B \in region$ computes the logical expression

$$\begin{aligned} lr\_inside(A,B) \stackrel{\text{def}}{=} \; & tp\_lr(14)(A,B) \;\vee\; tp\_lr(16)(A,B) \;\vee\; tp\_lr(26)(A,B) \;\vee \\ & tp\_lr(27)(A,B) \;\vee\; tp\_lr(29)(A,B) \;\vee\; tp\_lr(30)(A,B) \end{aligned}$$

The predicate *lr_inside* can now be employed in a query. For example, assuming the two relations `rivers(rname:string, route:line)` and `states(sname:string, area:region)`, we can pose the query "Determine river names and state names where the river is located within the state." as follows:

```
SELECT rname, sname FROM rivers, states WHERE route lr_inside area
```

A user can, of course, define arbitrary cluster predicates. It is especially possible to define different cluster predicates which do not completely exclude each other since they contain common basic topological predicates. To define a collection of mutually exclusive cluster predicates which covers all basic topological predicates, we allow the specification of *topological predicate clusters*. For example, each column of Table 13 represents such a predicate cluster. An extension of the DDL of SQL could formulate the predicate cluster for the point/line case in Table 13 as follows:

```
create tppredcluster pl_cluster
  (pl_disjoint(tp_pl(1), tp_pl(2)),
   pl_meet(tp_pl(3), tp_pl(4), tp_pl(5), tp_pl(6)),
   pl_inside(tp_pl(7), tp_pl(8), tp_pl(11), tp_pl(12))
   pl_overlap(tp_pl(9), tp_pl(10), tp_pl(13), tp_pl(14)))
```

Since the same cluster predicate can be defined differently in different predicate clusters, a user must be able to indicate which cluster she would like to use. A cluster can be selected by the DDL command

```
use tppredcluster pl_cluster
```

In the same way as a query language should be prepared to incorporate basic topological predicates, cluster predicates, and predicate clusters, these concepts should also be integrated into the application programming interface of a spatial database or GIS.

8 Conclusions and Future Work

From a formal and an application point of view, spatial applications require by far more complex geometric structures than the usual simple points, lines, and regions that can be currently found in spatial database systems, spatial query languages, and GIS. In the meantime, some GIS and database vendors have recognized this shortcoming and begun to incorporate more complex spatial data types into their systems. A first contribution of this paper is that we have defined very general and versatile spatial data types for complex points, complex lines, and complex regions in the two-dimensional Euclidean space on the basis of point set theory and point set topology. Complex points may be composed of a finite set of isolated points, complex lines may represent spatially embedded graphs possibly consisting of several connected components, and complex regions may consist of several components where each component possibly contains holes.

The introduction of complex spatial data types leads to a larger variety of topological relationships. The investigation and formalization of complete collections of mutually exclusive topological relationships between all combinations of complex spatial data types has been the second main contribution of this paper. It has been done on the basis of the well-known 9-intersection model.

Due to the large amounts of predicates for each type combination and the user's difficulty to handle them, our third contribution consists in the introduction of topological cluster predicates and topological predicate clusters. These two concepts allow the user to group basic topological predicates under a common name and thus to reduce the number of predicates.

A main topic of future work consists in the implementation of all basic topological predicates as well as cluster predicates. Literature on the implementation of topological predicates is rare (see Section 2). It is common view that a plane sweep technique [3] from Computational Geometry can solve the problem. But the implementation of a single algorithm for each topological predicate of each type combination can become rather troublesome. We pursue the idea to design a single evaluation algorithm for each type combination. The task of such an algorithm is to determine the topological relationship for a given scenario of two spatial objects. The determined predicate is then matched against the query predicate. The implementation will be part of SPAL2D which is a sophisticated *spatial algebra* under development for two-dimensional applications.

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