1. The data

We want to construct a curve segment $\mathbf{g}$ to match

- a feature point $\mathbf{y}^c$ (at (0,2)) exactly and
- at either end, second-order boundary data $\mathbf{y}^0$, a 2-jet, in the least-squares sense. Here the second-order data are represented as three Bézier points on a line segment.

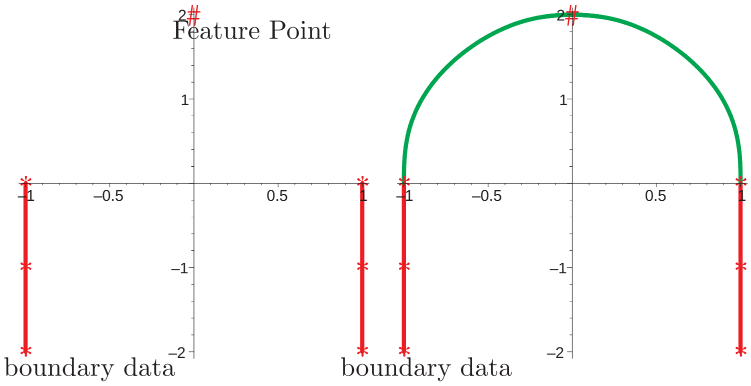

Figure 1: *left*: The data to be matched. *right*: approximate shape of a curve segment $\mathbf{g}$ fit to the data.

This segment will serve as *curve guide*. Note that the boundary data prescribe vertical tangents and zero curvature at $(\pm 1, 0, 0)$.

2. Guides consisting of a single polynomial

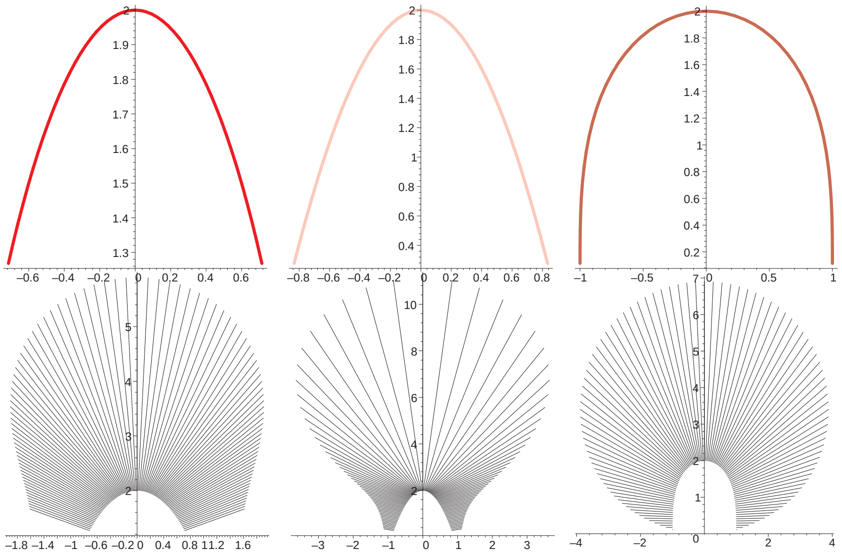

Figure 2: *top*: Fitting the data in 1 using a single cubic curve segment, a quartic and quintic fitting. *bottom*: curvature profile of each fit.

In addition to fitting the data, the guides **g** need to have a good curvature distribution. Here, we use 'hedge hog' plots that draw normals vectors scaled according to curvature.

3. Composite guides

We choose two C^2-connected cubic pieces to form $\mathbf{g}$ and approximate the boundary data. We parameterize the two pieces to have the parameter value $u = 0$ at the feature point, where they join. In Figure 3, *left*, we fit the boundary data by each piece at parameter $u = 1$. In Figure 3, *right*, however, we compose $\mathbf{g}$ with the map $\rho : \mathbb{R} \to \mathbb{R}$ defined as a quadratic with Bézier coefficients $1, 1.5, 3$ and then we fit the boundary data by the composed map $\mathbf{g} \circ \rho$ at parameter $u = 1$.

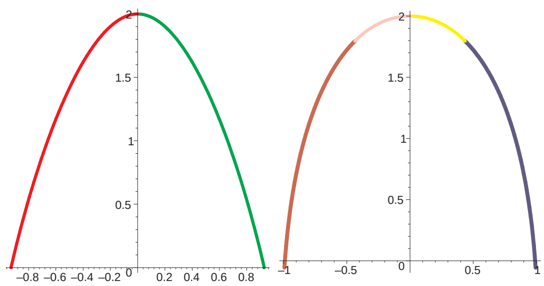

Figure 3: *left*: Two C^2 connected cubic pieces. *right*: Two C^2-connected reparamterized cubic pieces.

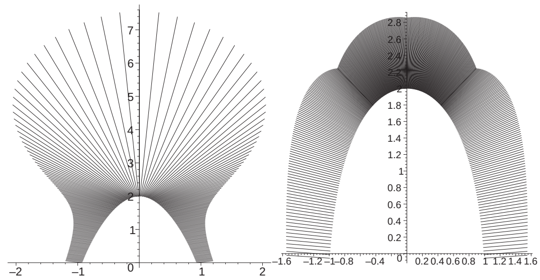

Figure 4: Curvature of the guides in Figure 3.

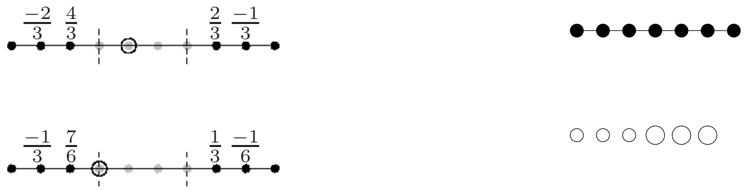

Figure 5: Hermit sampling to get quintic or C^2 cubics. (*left*) Interpolation stencil of a 2-jet at the corners to form C^2 cubics. (*right*) 2-jet at the corners to give a quintic curve.

4. Sampling the guide

Even the last piecewise cubic guide $\mathbf{g} \circ \rho$ did not fit the boundary data exactly. We will now *sample* $\mathbf{g} \circ \rho$ in a hermite fashion h to create an overall C^2 curve $\mathbf{x}^0 := h(\mathbf{g} \circ \rho)$ that matches the boundary data exactly, but leaves a gap between the left and the right piece.

We sample the guide curve $f := \mathbf{g} \circ \lambda^m \rho$ at $u = 0$ and at $u = 1$ to obtain three Bézier coefficients. The three Bézier coeeficents at each end alternatively define one degree 5 curve segment or three C^2-connected cubic curve segments (Illustrated in Figure 5). To match the boundary data $\mathbf{y}^0$ exactly, we replace the jet by the jet of the data.

We then fill the gap, by sampling again,

$$\mathbf{x}^m := h(\mathbf{g} \circ \lambda^m \rho), \quad \lambda := \frac{\rho(0)}{\rho(1)}.$$

The resulting sequence of curve pieces converges to $\mathbf{y}^c$ and formes and (inifinte) composite curve of degree 5.

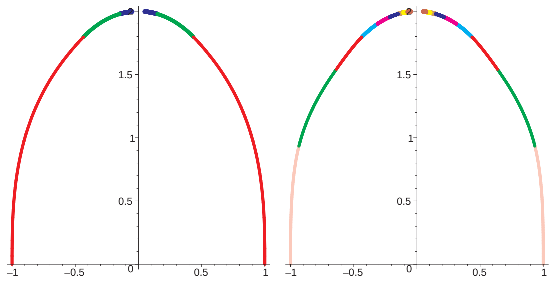

Figure 6: Hermit quintics and cubics fitting

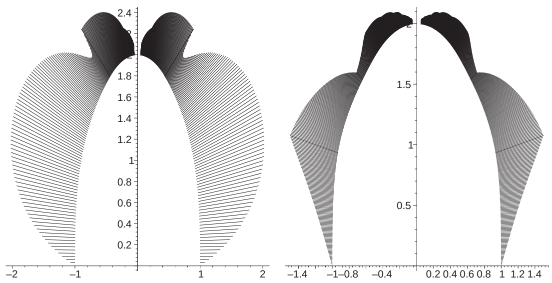

Figure 7: Curvature of Hermit quintics and cubics fitting

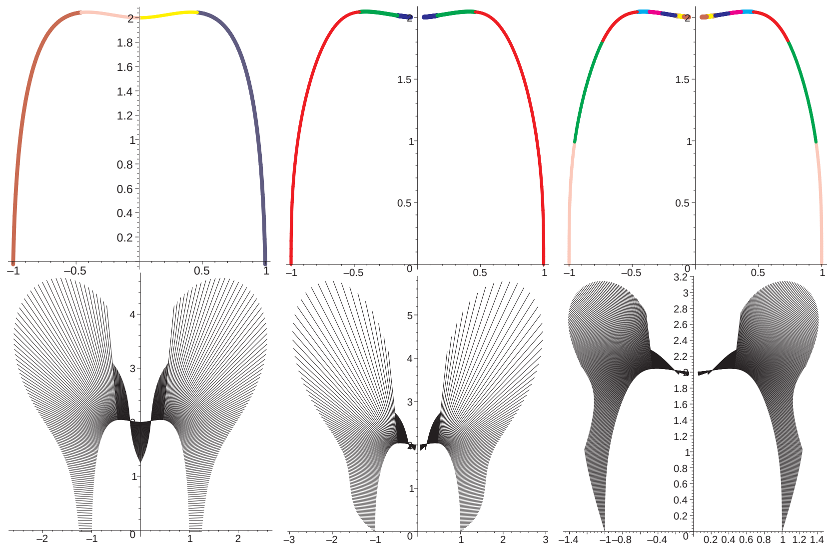

Figure 8: ct-map coefficients $1, 2, 3$. From *left* to *right*: C^2 composed cubics, Hermite sampled quintics and cubics. *bottom*: curvature comb.

5. The influence of parametrization on the guide shape

With different choices of ρ, the resulting fitting curves have different shapes. In the following figures, the map $\rho : \mathbb{R} \to \mathbb{R}$ defined as a quadratic with Bézier coefficients $1, 2, 3$ (Figure **??**), $1, 1.5, 2$ (Figure **??**), $1, 2, 4$ (Figure **??**) and $1, 4, 8$ (Figure **??**).

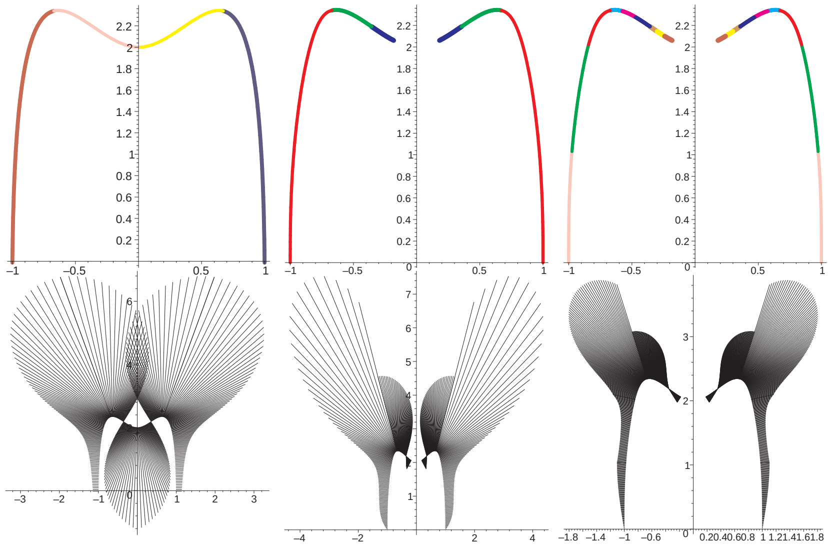

Figure 9: ct-map coefficients $1, 1.5, 2$. From *left* to *right*: C^2 composed cubics, Hermite sampled quintics and cubics.

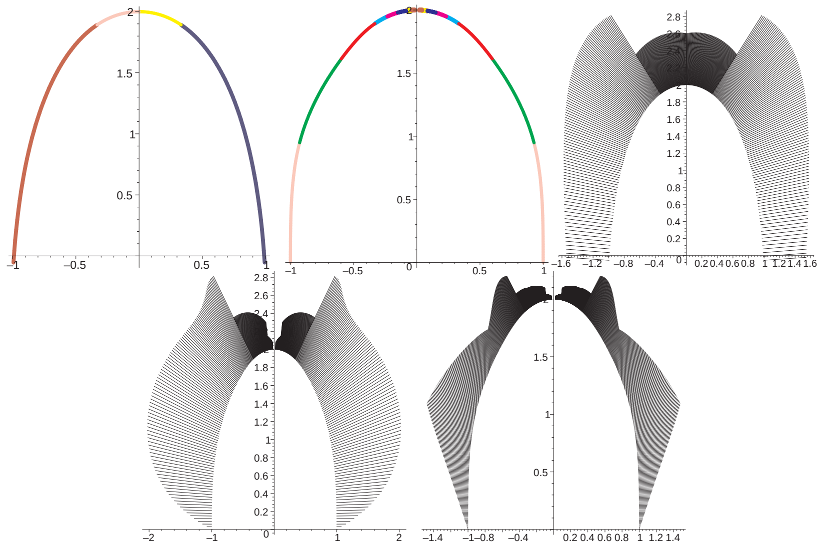

Figure 10: ct-map coefficients $1, 2, 4$. From *left* to *right*: C^2 composed cubics, Hermite sampled quintics and cubics.

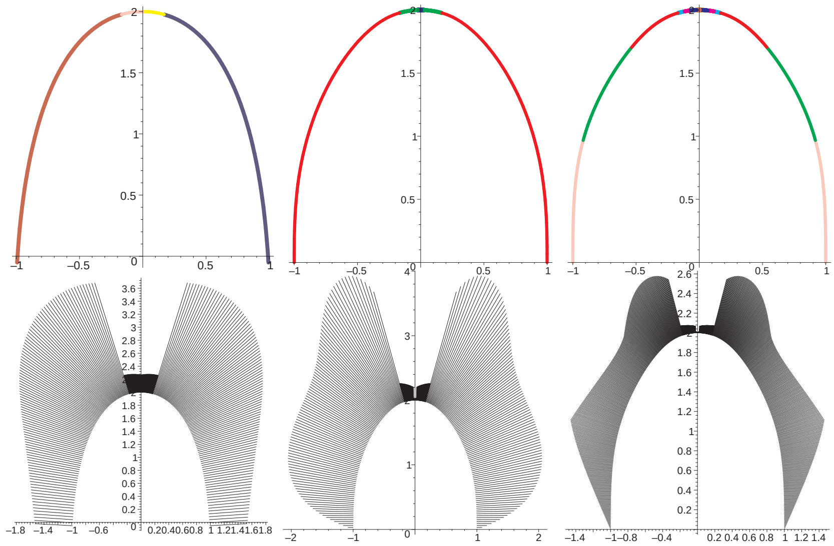

Figure 11: ct-map coefficients $1, 4, 8$. From *left* to *right*: C^2 composed cubics, Hermite sampled quintics and cubics.