A Practical Box Spline Compendium

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Abstract

Box splines provide smooth spline spaces as shifts of a single generating function on a lattice and so generalize tensor-product splines. Their elegant theory is laid out in classical papers and a summarizing book. This compendium adds a succinct but exhaustive survey of the important sub-space of symmetric low-degree box splines on symmetric lattices with special focus on two and three variables. Tables contrast the complexity in terms of support size and polynomial degree, analytic and reconstruction properties, and list available implementations and code.

1. Introduction

As a generalization of uniform polynomial tensor-product splines, and with the beautiful interpretation as a projection of a higher-dimensional box partition [1, 2, 3, 4], see Fig. 1, box splines have repeatedly commanded the attention of researchers seeking an elegant foundation for differentiable function spaces on low-dimensional lattices. No-



Figure 1: Box splines as a projection of (a) 2- and (b) 3-dimensional boxes [4]. Think of sand being dropped vertically, or of density under a vertical x-ray through uniform material.

tably, box splines provide the regular prototypes for generalized uniform polynomial subdivision algorithms [5, 6, 7] and have been advocated for reconstructing signals on non-Cartesian lattices, see Section 9. Where the repeating structure applies, the low total degree and support footprint, and the increased choice of isotropy compared to the tensor-product spline of compatible approximation order make box splines not only a viable, but a desirable alternative. For example, the four-direction 'Zwart Powell' (ZP) box spline M_{c11} on \mathbb{Z}^2 is C^1 of total degree 2, allowing the conic contours of any linear combination to be traced out as a smooth, rationally parameterized spline of degree two.

This compendium summarizes the current knowledge, with emphasis on d = 2 and d = 3 variables, for *symmetric* box spline spaces, i.e., box splines that have at least the symmetry of their domain lattice. The aim is to provide a succinct overview, via tables and illustrations, of the properties, literature and computational tools and code, and to characterize each box spline's efficiency in terms of smoothness, polynomial reproduction, support size and polynomial degree. That is, the compendium is intended to aid the applied computational mathematician in *choosing and using box splines*. For the many other beautiful aspects of box splines we recommend the seminal book [8].

2. Lattices and box splines

The goal of this section is to introduce the means and notation for a structured enumeration and definition of symmetric box splines: the relevant symmetries, the corresponding directions and their relation to box splines. For a general treatment of lattices and their symmetry groups, beyond the needs of this compendium, we refer to [9]. For detailed proofs of box spline properties, we recommend [8].

Lattices and Direction Sets. Given the integer grid \mathbb{Z}^d , any non-singular $d \times d$ generator matrix **G** defines a lattice $\mathbb{Z}_{\mathbf{G}} := \mathbf{G}\mathbb{Z}^d$. The symmetry group $\mathcal{SG}(\mathbb{Z}_{\mathbf{G}})$ of $\mathbb{Z}_{\mathbf{G}}$, represented as an orthogonal matrix group, consists of all orthogonal transformations that leave $\mathbb{Z}_{\mathbf{G}}$ invariant:

$$\mathcal{SG}\left(\mathbb{Z}_{\mathbf{G}}
ight) := \left\{ \mathbf{L} \in \mathbb{R}^{d imes d} : \mathbf{L}^T \mathbf{L} = \mathbf{I}_d ext{ and } \mathbf{L} \boldsymbol{j} \in \mathbb{Z}_{\mathbf{G}} ext{ for all } \boldsymbol{j} \in \mathbb{Z}_{\mathbf{G}}
ight\},$$

where \mathbf{I}_d is the $d \times d$ identity matrix. In the plane (2D) and 3-space (3D), five lattices are known for their high symmetries. They are listed in Table 1. To enumerate box splines, we collect the lattice direction vectors $\mathbf{j} \in \mathbb{Z}_{\mathbf{G}}$ into direction sets $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ consisting of one vector and its images under the symmetry group of the lattice. The index k is assigned by non-decreasing vector length, see Fig. 2 and Fig. 3, which is unique for $k \leq 3$, the cases of interest. (For k > 3, multiple direction sets can lie in the same spherical shell [9], e.g. (5,0) and (4,3) in \mathbb{Z}^2 .) Since $-\mathbf{j} = \mathbf{G}(-\mathbf{i})$ and $-\mathbf{i} \in \mathbb{Z}^d$ if $\mathbf{i} \in \mathbb{Z}^d$, for each $\mathbf{j} = \mathbf{G}\mathbf{i} \in \mathbb{Z}_{\mathbf{G}}$ also $-\mathbf{j} \in \mathbb{Z}_{\mathbf{G}}$, we list only one of \mathbf{j} and $-\mathbf{j}$ in $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ in order to avoid listing parallel directions

dim.	name	symbol	generator matrix	$\#\mathcal{SG}\left(\cdot ight)$
2	Cartesian	\mathbb{Z}^2	\mathbf{I}_2	8
4 -	hexagonal	\mathbb{Z}_{h}	$\mathbf{G}_{\mathrm{h}} := rac{1}{2} \left[egin{array}{c} 1 & 1 \ -\sqrt{3} & \sqrt{3} \end{array} ight]$	12
3	Cartesian	\mathbb{Z}^3	\mathbf{I}_3	48
	FCC (face-centered cubic)	$\mathbb{Z}_{ ext{fcc}}$	$\mathbf{G}_{\mathrm{fcc}} := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	48
	BCC (body-centered cubic)	$\mathbb{Z}_{ ext{bcc}}$	$\mathbf{G}_{\mathrm{bcc}} := \left[\begin{smallmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{smallmatrix}\right]$	48

Table 1: Five domain lattices for d = 2, 3. #S is the cardinality of the set S.



Figure 2: Stratifying 2D lattice points by distance to the origin \bullet such that each shell corresponds to a direction set $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ with $k = 1, k = 2, k = 3, k = 4, \ldots$



Figure 3: Stratifying 3D direction vectors in the (+, +, +) octant corresponding to direction sets $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ with k = 1, k = 2, k = 3. Table 2 lists coordinates.

Box Splines. Given a domain lattice $\mathbb{Z}_{\mathbf{G}}$, direction vectors $\boldsymbol{\xi} \in \mathbb{Z}_{\mathbf{G}}$ can be collected into a $d \times m$ direction matrix $\boldsymbol{\Xi}$ to define the centered box spline $M_{\boldsymbol{\Xi}}$ recursively, starting with the characteristic function $\chi_{\boldsymbol{\Xi}\boldsymbol{\Box}^d}$ on the half-open parallelepiped $\boldsymbol{\Xi}\boldsymbol{\Box}^d, \boldsymbol{\Box} := \left[-\frac{1}{2}, \frac{1}{2}\right)$, see [10, 8] and Fig. 4:

$$M_{\Xi} := \begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} M_{\Xi \setminus \boldsymbol{\xi}} \left(\cdot - t \boldsymbol{\xi} \right) \mathrm{d}t & \text{if } d < m, \ \boldsymbol{\xi} \in \Xi, \\ \frac{|\det \mathbf{G}|}{|\det \Xi|} \chi_{\Xi \Box^d} & \text{if } d = m \text{ and } \det \Xi \neq 0. \end{cases}$$
(1)



Figure 4: Construction of M_{c11} via successive directional convolutions along the directions (columns) of $[\Xi_{cc2} \ \Xi_{qc}]$ (see Table 2).

The centered box spline is invariant under exchange of columns and, up to rigid transformation, under multiplication of a column by -1: $M_{\Xi_1} = M_{\Xi_2}$ if and only if there exists a 'signed permutation' matrix **P** that can permute and/or change sign of a coordinate, such that $\Xi_1 = \Xi_2 \mathbf{P}$. Moreover, since for any linear map **L**, see [8, page 11],

$$M_{\Xi} = |\det \mathbf{L}| M_{\mathbf{L}\Xi}(\mathbf{L}\cdot), \tag{2}$$

many properties for centered box splines on the Cartesian lattice \mathbb{Z}^d transfer directly to $\mathbb{Z}_{\mathbf{G}}$ by a linear change of variables \mathbf{G} .

Let $\Xi \in \mathbf{G}\mathbb{Z}^{d \times m}$ with rank $\Xi = d$, M_{Ξ} the corresponding box spline, and $S_{\Xi} := \operatorname{span}(M_{\Xi}(\cdot - \mathbf{j}))$ the space of its shifts over the lattice $\mathbb{Z}_{\mathbf{G}}$. Then M_{Ξ} and S_{Ξ} have the following properties:

1. M_{Ξ} is non-negative and its shifts over $\mathbb{Z}_{\mathbf{G}}$ sum to 1: due to the factor $|\det \mathbf{G}|$ in (1)

$$\sum_{\boldsymbol{j}\in\mathbb{Z}_{\mathbf{G}}}M_{\boldsymbol{\Xi}}(\cdot-\boldsymbol{j})=1.$$

- 2. The support of M_{Ξ} is $\Xi \Box^d$, i.e., the centered set sum of the vectors in Ξ .
- 3. M_{Ξ} is a piecewise polynomial of total degree m d.

- 4. $M_{\Xi} \in C^{r-2}$. That is, M_{Ξ} is r-2 times continuously differentiable, where r is the minimal number of columns that need to be removed from Ξ to obtain a matrix whose columns do not span \mathbb{R}^d .
- 5. S_{Ξ} reproduces all polynomials of degree r 1.
- 6. The L^p approximation order of S_{Ξ} is r [8, page 61], i.e., for all sufficiently smooth f there exists a sequence $c : \mathbb{Z}_{\mathbf{G}} \to \mathbb{R}$ such that

$$\left\| f - \sum_{\boldsymbol{j} \in \mathbb{Z}_{\mathbf{G}}} c(\boldsymbol{j}) M_{\Xi}((\cdot - \boldsymbol{j})/h) \right\|_{p} = O(h^{r}), \quad h < 1.$$
(3)

- 7. S_{Ξ} forms a basis (the shifts are linearly independent) if and only if all square nonsingular submatrices of Ξ are unimodular, i.e., $|\det \mathbf{Z}| = 1$ for all $\mathbf{Z} \subset \Xi$ where $\mathbf{Z} \in \mathbb{R}^{d \times d}$ [8, page 41].
- 8. With vol $(\Xi \Box^d)$ denoting the volume of the support of M_{Ξ} , the number of coefficients on $\mathbb{Z}_{\mathbf{G}}$ required to evaluate a spline value is vol $(\Xi \Box^d) / |\det \mathbf{G}|$, [8, page 36].

The symmetry group of M_{Ξ} is defined analogous to the symmetry group of a lattice:

$$\mathcal{SG}(M_{\Xi}) := \left\{ \mathbf{L} \in \mathbb{R}^{d \times d} : \mathbf{L}^T \mathbf{L} = \mathbf{I}_d \text{ and } M_{\Xi} = M_{\Xi}(\mathbf{L} \cdot) \right\}.$$

A centered box spline M_{Ξ} on the domain lattice $\mathbb{Z}_{\mathbf{G}}$ is symmetric if it has the same or more symmetries than $\mathbb{Z}_{\mathbf{G}}$: $\mathcal{SG}(\mathbb{Z}_{\mathbf{G}}) \subset \mathcal{SG}(M_{\Xi})$. For example, the centered box spline defined by $\Xi := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not symmetric: its symmetry group is $\{\mathbf{I}_2, -\mathbf{I}_2\}$, but the symmetry group of \mathbb{Z}^2 has the cardinality 8 of the signed permutation group. If $\boldsymbol{\xi} \in \mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ is a column of Ξ then all directions of $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$ must be columns in Ξ to make M_{Ξ} symmetric. This can be seen as follows. For any $\boldsymbol{\xi} \in \mathbb{Z}_{\mathbf{G}}$, let $\Xi := \{\mathbf{L}\boldsymbol{\xi} : \mathbf{L} \in \mathcal{SG}(\mathbb{Z}_{\mathbf{G}})\}$. Then for any $\mathbf{L} \in \mathcal{SG}(\mathbb{Z}_{\mathbf{G}})$, the set of directions Ξ equals the set $\mathbf{L\Xi}$ and $|\det \mathbf{L}| = 1$ so that by (2) $M_{\Xi} = |\det \mathbf{L}|M_{\mathbf{L\Xi}}(\mathbf{L}\cdot) = M_{\Xi}(\mathbf{L}\cdot)$. That is, M_{Ξ} is symmetric. It suffices to include either $\boldsymbol{\xi}$ or $-\boldsymbol{\xi}$ into Ξ since for any $\boldsymbol{\xi} \in \mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, k)$

$$\int_{-1/2}^{1/2} f(\cdot - t\boldsymbol{\xi}) dt = \int_{-1/2}^{1/2} f(\cdot - t(-\boldsymbol{\xi})) dt = \int_{0}^{1/2} f(\cdot - t\boldsymbol{\xi}) dt + \int_{0}^{1/2} f(\cdot - t(-\boldsymbol{\xi})) dt.$$

3. Choice of direction vectors

The algebraic and differential geometric properties of Section 2 imply that the efficiency of a box spline space is closely related to the choice of direction vectors in the construction of the box spline and favors the vectors to be

- snapped to the lattice: this allows the approximation order to be maximal. By contrast, the shifts of $M_{[1/2]}$ on \mathbb{Z} , whose directions do not snap to the lattice, do not sum to 1. And $M_{[1,1/2]}$, whose shifts on \mathbb{Z} sum to 1, form a spline space $S_{[1,1/2]}$ that has intervals where the spline is constant and cannot match linear functions.

- short: since longer vectors result in larger support and more vectors are required to achieve symmetry, increasing the degree.
- uniformly distributed: for the same degree, uniformity increases the continuity and approximation order. For example, see Table 3, the bi-linear B-spline M_{c20} and the ZP element M_{c11} have degree 2, but both the continuity and the approximation order of M_{c11} are higher by one than those of M_{c20} .
- in $\mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, 1)$: for the five lattices, direction sets with k > 1 yield Ξ that are not unimodular, and so the box spline shifts are not linearly independent [8].

Uniform distribution on a lattice is in competition with shortness since equi-distribution of directions requires inclusion of farther lattice points.

Table 2: The direction sets of the five domain lattices in Table 1: repeating directions are grayed out. Numbers in the parentheses denote the cardinality of corresponding direction set, cf. Fig. 2 and Fig. 3.

lattice	k = 1		$\mathcal{DS}\left(\mathbb{Z}_{\mathbf{G}},k\right)$ $k = 2 \qquad \qquad k = 3$			k = 4		
\mathbb{Z}^2 $\mathbb{Z}_{ ext{h}}$ \mathbb{Z}^3	$egin{array}{c} \Xi_{ m cc2} \ G_{ m h}\Xi_3 \ \Xi_{ m cc3} \end{array}$	(2) (3) (3)	$\mathbf{G}_{\mathrm{h}} \begin{bmatrix} \boldsymbol{\Xi}_{\mathrm{qc}} \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix} \\ \boldsymbol{\Xi}_{\mathrm{fcc}} \\ \mathbf{\Xi}_{\mathrm{fcc}} \end{bmatrix}$	(2) (3) (6)	$\begin{array}{c} 2\Xi_{cc2} \\ \mathbf{G}_{h}(2\Xi_{3}) \\ \mathbf{\Xi}_{bcc} \end{array}$	(2) (3) (4)	$ \begin{array}{c} \{\pi(2,\pm 1)\} \\ \mathbf{G}_{\mathrm{h}} \begin{bmatrix} 1 & 2 & 3 & 3 & 2 \\ 3 & 3 & 2 & 1 & -1 & -2 \end{bmatrix} \\ 2 \Xi_{\mathrm{cc3}} \end{array} $	(4) (6) (3)
$\mathbb{Z}_{ ext{fcc}}$ $\mathbb{Z}_{ ext{bcc}}$	$\mathbf{\Xi}_{ ext{fcc}} \ \mathbf{\Xi}_{ ext{bcc}}$	(6) (4)	$2 \Xi_{cc3}$ $2 \Xi_{cc3}$	$\begin{array}{c} (3) \\ (3) \end{array}$	$\{\pi(2,\pm 1,\pm 1)\}$ 2 $\Xi_{\rm fcc}$	(12) (6)	$2\Xi_{ m fcc} \ \{\pi(3,\pm1,\pm1)\}$	(6) (12)

 $\{\pi(x_1, x_2, \dots, x_d)\}$ is the set of vectors generated by permuting the coordinates x_i . E.g. $\{\pi(2, \pm 1)\} = \{(2, 1), (2, -1), (1, 2), (-1, 2)\}$

Table 2 lists the direction sets for the bivariate and trivariate domain lattices of Table 1 in terms of the matrices (see Fig. 2 and 3):

where the subscripts are to remind of Cartesian (cc2, cc3) quincunx (qc), 3 directions, FCC, and BCC directions, respectively.

4. Bivariate box splines

Since the third direction set in Table 2 of \mathbb{Z}^2 and \mathbb{Z}_h already repeat the first, we restrict the list of bivariate box splines in Table 3 to $\mathcal{DS}(\mathbb{Z}_G, k)$ for k < 3, as illustrated



Figure 5: Directions (arrows) and supports (polygons with black edges) of select bivariate box splines with polynomial pieces delineated by knot lines (gray lines).

Table 3: Bivariate symmetric box splines up to degree 6. M_{cn0} is the tensor-product B-spline, M_{c11} is the Zwart-Powell (ZP) element, M_{c21} is the extended 6-direction ZP element, and M_{h10} is the hat function. The continuity is C^{r-2} with r defined by Property 4 of Section 2. Some splines lack reference since they have not been investigated, likely due to their large stencil size.

lattice	dire 1	ection sets 2	degree	differentiability $r-2 =$	stencil size	reference
	n	0	2n - 2	n-2	n^2	[11]
	1	1	2	1	7	[12, 13, 14]
\mathbb{Z}^2	2	1	4	2	14	[15, 16, 17]
	3	1	6	3	23	
	2	2	6	4	28	
F77	n	0	3n - 2	2n - 2	$3n^2$	[18, 19, 20, 17]
⊿ _h	1	1	4	3	24	

in Fig. 5. We could skip k = 3 and consider the box spline defined by $\bigcup_{k=1,2,4} \mathcal{DS}(\mathbb{Z}^2, k)$ with 2 + 2 + 0 + 4 = 8 directions, but the corresponding box spline has a large support

and degree 8-2=6, while the resulting C^5 continuity is unlikely to match any generic application needs. Similarly, the box spline defined by $\mathcal{DS}(\mathbb{Z}^2, 4)$ yields a box spline of degree 2with support size 24, whereas the ZP spline M_{c11} has the same smoothness but support size 7.



Figure 6: Directions and supports of select trivariate box splines.

Denoting by n_k the number of repetitions of the kth direction set, the box spline on \mathbb{Z}^2 are named $M_{cn_1n_2}$ and those on \mathbb{Z}_h are named $M_{hn_1n_2}$. Table 3 leaves out direction sets of the form (0, n) and (1, n) for \mathbb{Z}^2 , since their properties do not improve on (n, 0) and (n, 1), respectively and result in a larger support. Analogously, (0, n) is omitted for \mathbb{Z}_h . We note that the options for C^1 continuity are M_{c30} (9) and M_{c11} (7), with the stencil sizes listed in parentheses. For C^2 continuity the options are M_{c40} (16), M_{c21} (14), and M_{h20} (12). The only linearly independent symmetric box splines are M_{cn0} ,

1.44:00	direction sets			differentiability	stencil	note	
lattice	1	2	3	degree	r - 2 =	size	/ reference
	n	0	0	3n - 3	n-2	n^3	B-splines [21]
	1	1	0	6	3	87	[22]
	2	1	0	9	4	172	
	1	0	1	4	2	53	[23, 24, 25, 26]
\mathbb{Z}^3	1	0	2	8	4	249	
	2	0	1	7	4	120	
	0	n	0	6n - 3	3n - 2	$32n^3$	[27]
	0	1	1	7	5	216	
	0	0	n	4n - 3	2n - 2	$16n^{3}$	
	n	0	0	6n - 3	3n - 2	$16n^{3}$	[28, 29]
	1	1	0	6	3	86	$[22]^{\dagger}$
$\mathbb{Z}_{ ext{fcc}}$	1	2	0	9	4	228	
	0	n	0	3n - 3	n-2	$4n^3$	B-splines
	0	0	1	9	7	784	
	n	0	0	4n - 3	2n - 2	$4n^3$	[30, 4]
	2	1	0	8	4	106	
	1	1	0	4	2	30	[31]
	1	2	0	7	4	92	
$\mathbb{Z}_{ ext{bcc}}$	1	0	1	7	5	200	
	0	n	0	3n - 3	n-2	$2n^3$	B-splines [32]
	0	1	1	6	3	174	
	0	2	1	9	4	344	
	0	0	n	6n - 3	3n - 2	$64n^{3}$	

Table 4: Trivariate symmetric box splines up to degree 9. Note that many splines have not been investigated, likely due to their large stencil size.

[†] The box spline proposed in [22] is a sibling of M_{f110} built from the direction matrix $\begin{bmatrix} \Xi_{fcc} & \Xi_{cc3} \end{bmatrix}$. Since Ξ_{cc3} do not snap to \mathbb{Z}_{fcc} , the resulting approximation order is lower than M_{f110} (but M_{f110} has a larger support).

i.e., the B-splines on \mathbb{Z}^2 , and M_{hn0} on \mathbb{Z}_h . Other linearly independent box splines, such as the three-direction box spline on \mathbb{Z}^2 [33], are not symmetric. The stencil size explains why several box splines have not been investigated in detail.

5. Trivariate box splines

Analogous to the bivariate case, denoting by n_k the number of repetitions of the kth direction set, the box splines on \mathbb{Z}^3 , \mathbb{Z}_{fcc} , and \mathbb{Z}_{bcc} are named

$M_{cn_1n_2n_3}, M_{fn_1n_2n_3}, \text{ and } M_{bn_1n_2n_3}$

in Table 4. Fourth direction vectors are not used since, e.g. for M_{b*} , they are typically too long and too many. While there are asymmetric box splines whose shifts are linearly independent, e.g. the four-direction box splines on \mathbb{Z}^3 , the only symmetric linearlyindependent box splines are M_{cn00} , the B-splines on \mathbb{Z}^3 , M_{fn00} on \mathbb{Z}_{fcc} , and M_{bn00} on \mathbb{Z}_{bcc} . That is M_{*n00} are the only symmetric box splines that form a basis. Listing the support sizes in parentheses, the C^1 box splines are M_{c300} (27), M_{c010} (32), M_{f100} (16), M_{f030} (108), M_{b030} (54), M_{b001} (64) and the C^2 box splines are M_{c400} (64), M_{c101} (53), M_{c002} (128), M_{f040} (256), M_{b200} (32), M_{b110} (30), M_{b040} (128). Due to their small degrees, listed in angle brackets, and supports M_{c300} (6), M_{c010} (3), M_{f100} (3) (see Fig. 4) stand out as efficient for C^1 and M_{b200} (5), M_{b110} (4) for C^2 .

6. Multi-variate box splines

The five lattices in two and three variables are instances of *d*-dimensional lattices, d > 3 whose detailed definition can be found in [4, 34]. The generator matrices of the four lattices other than \mathbb{Z}^d [9] are as follows:

$$\mathbf{A}_{d} := \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix}, \mathbf{A}_{d}^{*} := \frac{1}{d+1} \begin{bmatrix} d & -1 & \cdots & -1 & -1 \\ -1 & d & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & d & -1 \\ -1 & -1 & \cdots & -1 & d \\ -1 & -1 & \cdots & -1 & -1 \end{bmatrix},$$
$$\mathbf{D}_{d} := \begin{bmatrix} -1 & 1 & & & \\ -1 & -1 & 1 & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}, \text{ and } \mathbf{D}_{d}^{*} := \begin{bmatrix} 1 & & 1/2 \\ 1 & & 1/2 \\ & & & 1/2 \\ & & & 1/2 \end{bmatrix}.$$

Note that \mathbf{A}_d and \mathbf{A}_d^* are $(d+1) \times d$ and the corresponding lattices are generated in the hyperplane defined by $x_1 + \cdots + x_{d+1} = 0$ in \mathbb{R}^{d+1} .

Table 5 lists the first and second direction sets of the five lattices. As in the bi- and the trivariate cases, various symmetric box splines can be constructed from these directions. We observe that for \mathbb{D}_d^* , d > 4, there is a rich set of first directions, all corresponding to B-splines, to build smooth symmetric splines. Table 6 lists some important classes of

Table 5: The first and second direction sets of the five main lattices where $\mathbb{A}_d := \mathbf{A}_d \mathbb{Z}^d$, $\mathbb{A}_d^* := \mathbf{A}_d^* \mathbb{Z}^d$, $\mathbb{D}_d := \mathbf{D}_d \mathbb{Z}^d$, and $\mathbb{D}_d^* := \mathbf{D}_d^* \mathbb{Z}^d$. For ease of notation, opposite directions $\{\pm \mathbf{j} : \mathbf{j} \in \mathcal{DS}(\mathbb{Z}_G, k)\}$ are enumerated and the directions of \mathbb{A}_d^* are scaled by (d+1) and those of \mathbb{D}_d^* by 2. As in [9], a^{α} abbreviates α -fold repeating entries a, \ldots, a .

lattice	dim.	k = 1	k = 2
\mathbb{Z}^d	$d \ge 2$	$\{\pi(\pm 1, 0^{d-1})\}$	$\{\pi((\pm 1)^2, 0^{d-2})\}$
\mathbb{A}_d	$\begin{array}{c} d{=}2\\ d{>}2 \end{array}$	$ \{ \pi(1, -1, 0) \} \\ \{ \pi(1, -1, 0^{d-1}) \} $	$\{\pm \pi(2,-1,-1)\}\ \{\pi(1^2,(-1)^2,0^{d-4})\}$
\mathbb{A}_d^*	$d \ge 2$	$\{\pm\pi(d,(-1)^d)\}$	$\{\pm\pi((d-1)^2,(-2)^{d-1})\}$
\mathbb{D}_d	$\begin{array}{c} d=3\\ d>3 \end{array}$	$ \{ \pi(\pm 1, \pm 1, 0) \} \\ \{ \pi((\pm 1)^2, 0^{d-2}) \} $	$ \{ \pi(\pm 2, 0, 0) \} $ $ \{ \pi(\pm 2, 0^{d-1}) \} \cup \{ \pi((\pm 1)^4, 0^{d-4}) \} $
\mathbb{D}_d^*	$\begin{array}{c} d{=}2,3 \\ d{=}4 \\ 4{<}d{<}8 \\ d{=}8 \\ d{>}8 \end{array}$	$ \{ ((\pm 1)^d) \} \{ ((\pm 1)^4) \} \cup \{ \pi (\pm 2, 0^3) \} \{ \pi (\pm 2, 0^{d-1}) \} \{ \pi (\pm 2, 0^{d-1}) \} \{ \pi (\pm 2, 0^{d-1}) \} $	$ \{ \pi(\pm 2, 0^{d-1}) \} \\ \{ \pi((\pm 2)^2, 0^2) \} \\ \{ ((\pm 1)^d) \} \\ \{ ((\pm 1)^d) \} \cup \{ \pi((\pm 2)^2, 0^{d-2}) \} \\ \{ \pi((\pm 2)^2, 0^{d-2}) \} $

Table 6: Select box splines for d > 3. Shifts of the box splines for \mathbb{Z}^d , \mathbb{A}_d and \mathbb{A}_d^* yield a basis. $\mathcal{DS}(\mathbb{Z}^d, 2)$ are box splines on \mathbb{D}_d .

lattice	dim.	direction 1	sets	degree	differentiability $r-2 =$	reference
\mathbb{Z}^d	$d \ge 2$	n	0	d(n-1)	n-2	B-splines [11]
\mathbb{A}_d	$d \ge 2$	1	0	$d(d{-}1)/2$	$d{-}2$	[34]
\mathbb{A}_d^*	$d \ge 2$	n	0	(d+1)n-d	2(n-1)	[4]
\mathbb{D}_d	$d \ge 2$	$d(d{-}1)$	0	d(d-2)	$2d{-}4$	[34]
	d=4	1^{\dagger}	0	4	2	[35]
Ш) *	d=4	$1^{\dagger\dagger}$	0	8	4	[34]
\mathbb{D}_d	$5 \leq d \leq 7$	1	1	2^{d-1}	2^{d-2}	[34]
	d > 4	n	0	d(n-1)	n-2	B-splines

[†] Constructed from directions $\{((\pm 1)^4)\}$ only.

^{††} Constructed from directions $\{\pi(\pm 2, 0^3)\}$ and $\{((\pm 1)^4)\}$.

box splines whose shifts live on these high-dimensional lattices, see e.g. [34]. Note that for some dimensions, two different direction sets share the same distance: for \mathbb{D}_4^* there

are 16/2 + 8/2 = 12 first directions of the patterns $(\pm 1, \pm 1, \dots, \pm 1)$ and $\pi(\pm 2, 0, 0, 0)$ and either or both groups yields a symmetric box spline.



Figure 7: The polynomial pieces in the support of M_{c11} and the BB-net (scaled by 8).



Figure 8: From [15]. (a) The polynomial pieces in the support of M_{c21} . Pieces of the same color have the same BB-net after appropriate rigid transformation and the BB-nets (multiplied by 192) of the pieces labeled b,...,h are shown in (b)–(h).

7. Conversion to piecewise polynomial form

It is useful to express the box spline pieces as polynomials, and in particular in the Bernstein-Bézier (BB-) form, see e.g. [36]. The partition into pieces follows from the convolution directions. The BB-coefficients are obtained from the differentiability constraints across boundaries and by normalizing the map, see [17]. Fig. 7, Fig. 8 and Fig. 9 show examples of the re-representation in BB-form. For trivariate box splines, using the constraints can be error-prone. An easier approach is to sample the spline at sufficiently many interior points, using one of [37, 38], and solve for the BB-coefficients, keeping in mind that the coefficients are integers after scaling by a known multiple, see [39]; or, and this is faster and yields polynomial pieces in partially factored form, to apply a Green's function decomposition and inverse Fourier transform [40].



Figure 9: From [15]. (a) The polynomial pieces in the support of M_{h20} . Pieces of the same color have the same BB-net after appropriate rigid transformation and the BB-nets (multiplied by 24) of the pieces labeled b,c,d are shown in (b),(c),(d).

8. Efficient evaluation

By reversing the convolution, the algorithms of [37, 38] evaluate box splines recursively. This process is stable except near the boundaries between the polynomial pieces, namely the knot lines in 2D and the knot planes in 3D. Near boundaries, de Boor [37] applies a random perturbation and [38] propose careful bookkeeping. Converting the box spline pieces to BB-form yields much faster and stable evaluation [39], also of derivatives. A general technique to accelerate evaluation is to leverage symmetry [41, 35] with a general implementation available at [35] that automates steps and generates GPU kernels. Table 7 lists box splines with an available optimized evaluation code, some implemented on the GPU for high parallelism.

box spline	algorithm	code
M_{c400}	[42, 43]	[42]
$M_{\rm c010}$	[27]	[27]
M_{c101}	[26]	[26]
M_{f100}	[28, 29]	[29]
$M_{\rm b200}$	[44, 45]	[45]
$M_{\rm b110}$	[31, 46, 41]	[47]
$M_{ m b040}$	[32]	

Table 7: Some fast 3D box spline evaluation implementations. See also [35].

Subdivision offers a stable and fast alternative when rendering an approximation, say a triangulation of a bivariate box spline graph. An alternative approximate evaluation is based on Fast Fourier Transform [48].

lattice	a.o.	box spline	$24q_{0}$	$-12q_{1}$	references
\mathbb{Z}^2	3	M_{c30}, M_{c11}	18	3	[34]
	4	$M_{\rm c20}, M_{\rm c21}$	40	4	
\mathbb{Z}_{h}	4	$M_{\rm h20}$	13	2	[4]
	9	M_{c300}	21	3	
2	3	$M_{ m c010}$	24	4	[27]
\mathbb{Z}^{3}	4	M_{c400}	24	4	
		M_{c101}	27	5	[34]
		$M_{ m c002}$	36	8	
	9	$M_{ m f100}$	18	1	[34]
$\mathbb{Z}_{ ext{fcc}}$		$M_{ m f030}$	30	3	
	4	$M_{ m f040}$	36	4	
	2	$M_{ m b030}$	24	3	
Zihaa		$M_{ m b001}$	28	4	
	4	$M_{\rm b200}, M_{\rm b110}$	20	2	[49, 4, 31]
	4	$M_{ m b040}$	28	4	-

Table 8: Quasi-interpolants of select box splines of approximation order (a.o.) 3 or 4. Note that q_0 and q_1 are scaled for clearer presentation. Entries without reference are newly derived for completeness.

9. Use for reconstruction or approximation

A promising application of box splines is the approximation and reconstruction of a function f from samples $\{f(\mathbf{j}) : \mathbf{j} \in \mathbb{Z}_{\mathbf{G}}\}$ on a lattice $\mathbb{Z}_{\mathbf{G}}$. To attain the maximal approximation order of the box spline space, i.e., to obtain c in Eq. (3), the samples are convolved with a discrete *quasi-interpolant* to form the control points

$$c(\boldsymbol{j}) := q_0 f(\boldsymbol{j}) + q_1 \sum_{\boldsymbol{k} \in \mathcal{DS}(\mathbb{Z}_{\mathbf{G}}, 1)} \left(f(\boldsymbol{j} + \boldsymbol{k}) + f(\boldsymbol{j} - \boldsymbol{k}) \right), \quad \forall \boldsymbol{j} \in \mathbb{Z}_{\mathbf{G}}$$

of the optimally approximating spline $\sum_{j \in \mathbb{Z}_{\mathbf{G}}} c(j) M(\cdot - j)$. Several techniques exist to derive quasi-interpolants for box splines [21, 8, 50, 51]. Table 8 lists quasi-interpolants, defined by q_0 and q_1 , for the box splines of approximation order 3 or 4 of Table 3 and Table 4. Level sets of quasi-interpolating functions in three variables are used to display Computed Tomography (CT) and Magnetic Resonance Imaging (MRI) data. A standard test function is the Marschner-Lobb signal [52], a combination of Dirac pulses and a circularly symmetric, disc-shaped component, see Fig. 10(h). Fig. 10 compares how convolution directions enhance or prevent reproduction of the circular features.



Figure 10: Ray-intersection rendering (ray-casting) of a level set of the Marschner-Lobb signal (h) with identical sampling density on their domain lattices.

10. Splines from pieces and unions of boxes

One can consider the characteristic function of a piece of the box or of a union of boxes, and then convolve these characteristic functions. Convolving the characteristic function of half of a box in 2D, i.e., of a triangle, yields *half-box spline* spaces with properties akin to box splines [53, 54, 55, 3, 56]. Alternatively, one can juxtapose non-centered boxes to form the Voronoi cell of a lattice, i.e., the region nearest to a lattice point. The convolution of the characteristic function of the Voronoi cell then yields *Voronoi splines* [57, 58]. Voronoi splines provide an example of how asymmetric splines can be linearly combined to form symmetric splines. Note though that such splines typically do not yield nested spaces [59].

11. Conclusion

Symmetric box splines provide a mature and powerful framework for shift-invariant smooth functions on a lattice. For bi- and tri-variate splines, a number of efficient box splines are now well-documented and come with optimized implementations.

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