C^2 Free-Form Surfaces of degree $(3, 5)$

Jörg Peters

November 30, 2001

Abstract

This paper introduces new techniques for modeling low degree, smooth free-form surfaces of unrestricted patch layout. In particular, surfaces that are C^2 after reparametrization can be built from tensor-product Bézier or spline patches of degree $(3, 3)$ and (3,d+2); at extraordinary points, these surfaces have the flexibility of C^2 splines of total degree $d > 0$. The particular choice, $d = 3$, yields more than $\mathtt{n} + 5$ vector-valued degree of freedom where $\mathtt{n}$ patches join. The techniques generalize to G^k constructions of free-form surfaces of degree $(k+1, d+2k-2)$.

1 Introduction

High quality surfaces require continuity of curvatures, while efficient computation, say of intersections, favors polynomial representations of low degree. In practice, curved shapes with a tensor-product layout are therefore typically designed using splines of degree $(3, 3)$ or $(5, 5)$. For arbitrary patch layout, however, known surface constructions of degree $(3, 3)$ are only tangent continuous (see e.g. [GZ94, Pet94, Rei95, Pet00]); and it seems unlikely that curvature continuity can be achieved using just bicubic patches except if one is willing to accept shape deficiencies such as flat regions. Even if we use an infinite sequence of bicubic patches near the extraordinary points, as in the Catmull-Clark generalized subdivision scheme [CC78], we obtain at best bounded curvature [Sab91]. The new techniques yield in particular curvature continuous surfaces consisting of splines of degree $(3, 3)$ plus some strips of splines of degree $(3, 5)$.

With the exception of [GZ99], curvature continuous polynomial constructions using tensor-product patches are of degree $(9, 9)$ or higher [Hah89, GH89, Ye97] or they use an approach developed by Prautzsch and Reif [Pra97, Rei98]: a disk-shaped region in $\mathbb{R}^2$ defined by a C^k tensor-product spline of degree $(k+1, k+1)$ is mapped onto a polynomial surface of total degree d in $\mathbb{R}^3$ to cover the neighborhood of an extraordinary point by a surface of degree $(dk+d, dk+d)$. If $k = 2$ and $d = 2$, the degree is as low as (6,6); but, at higher-order saddle points, $d = 2$ leads to undesirable flat patches rather than zero Gaussian curvature at the central point and negative Gaussian curvature in the immediate neighborhood (cf. Figure 9,*right*). Quadratic boundary curves in [GZ99] lead to similar shape deficiencies. Increasing d to 3, as suggested in [Rei99] to avoid flat patches, leads again to patches of degree $(9, 9)$. [PU00, BR97, Pet96] explain constructions using three-sided patches, the last as low as total degree 8. C^2 surfaces

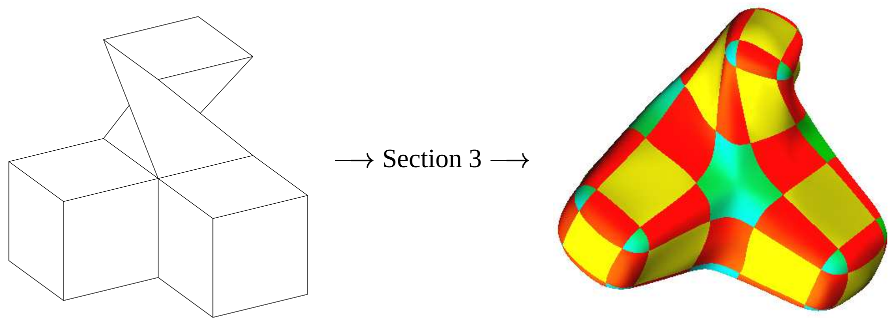

Figure 1: Control structure and corresponding free-form spline surface.

can be constructed from the ideas in [GH95, DL89, NG00] using rational patches of degree $(9, 9)$ and higher. [Du88, Her96] use second order 'Gregory patches' of degree $(5, 5)$ that are singular in the corners and have more than twice the number of coefficients of the construction in Section 3 below.

Outline. Section 2 develops the algebraic foundation of the new approach. In particular, Theorem 1 addresses the transition from an extraordinary point surrounded by $\mathtt{n}$ patches (cf. the 6-sided region on the right of Figure 1 corresponding to the 6-valent node on the left) to the regular tensor-product patch structure (*dark grey* and *light grey* regions). Section 3 sketches an algorithm for creating a C^2 surface that consists of polynomial patches of maximal degree $(3, d+2)$ and follows the outlines of a polyhedral mesh (cf. Figure 1); the neighborhoods of extraordinary points are modeled as C^2 splines of total degree d. Section 4 summarizes the new ideas and shows how the construction generalizes to kth order continuity.

1.1 Notation, definitions and concepts.

Definition 1 *With* $\mathtt{e}_1 := \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ *and* $\mathtt{e}_2 := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, *the* i*th partial derivative of a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *at the point* $u \in \mathbb{R}^2$ *is given as*

$$D_i f(u) := \lim_{t \to 0} \frac{f(u + t\mathtt{e}_i) - f(u)}{t}.$$

We also define $D^0 f := f$, $Df := [D_1 f, D_2 f]$ *and* $D^{k+1} f := DD^k f$ *for* $k \geq 1$. *Let* e *parametrize a line segment in* $\mathbb{R}^2$. *Then we denote the derivative in a direction* $\mathtt{e}^\perp$ *perpendicular to* e *by* $D_{\mathtt{e}^\perp} f$. *If* $D^\kappa_{\mathtt{e}^\perp} f$ *restricted to* e *is a (univariate) polynomial then*

$$d^\kappa_{f,e} := \textit{degree}\left(D^\kappa_{\mathtt{e}^\perp} f(e)\right).$$

To reduce the number of parentheses, we follow the convention that composition $\circ$ precedes differentiation $D_i f$, precedes the composition $f(e)$ and evaluation of a multilinear form $f\langle\cdot\rangle$. The latter is used for combining the multivariate chain rule and product rule for $D^\kappa(\mathbf{g} \circ \mathbf{r})$ in Faá di Bruno's Law (see Appendix).

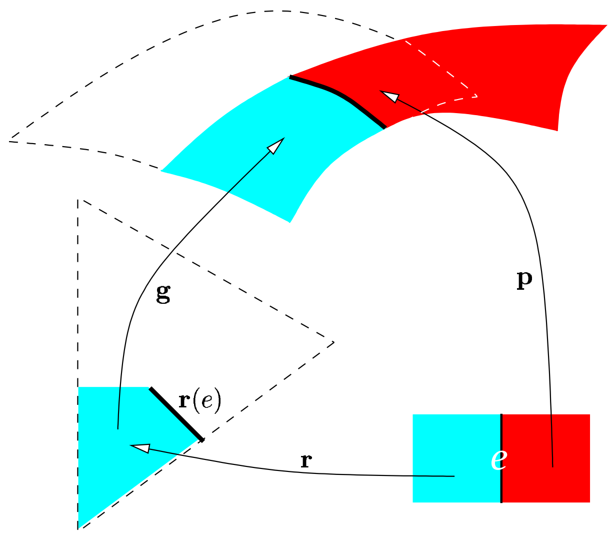

Figure 2: Reparametrization $\mathbf{r}$ and geometry maps $\mathbf{p}$ and $\mathbf{g}$. For a G^k join the κth transversal derivatives of $\mathbf{g} \circ \mathbf{r}$ and $\mathbf{p}$ have to agree along e.

Definition 2 *A* subdomain *is a simple, closed subset of* $\mathbb{R}^2$*, bounded by a finite number of regularly parametrized* edges*. Let* $e_{\mathbf{p}}$ *parametrize an edge of the subdomain* $\mathbb{D}_{\mathbf{p}}$ *and* $e_{\mathbf{g}}$ *an edge of the subdomain* $\mathbb{D}_{\mathbf{g}}$*. Denote an open neighborhood of a set* X *by* $\mathcal{N}(X)$*. Then* $\mathbf{r} : \mathcal{N}(e_{\mathbf{p}}) \to \mathcal{N}(e_{\mathbf{g}})$ *is a* C^k reparametrization *between* ($\mathbb{D}_{\mathbf{p}}$,$e_{\mathbf{p}}$) *and* ($\mathbb{D}_{\mathbf{g}}$,$e_{\mathbf{g}}$) *if it maps* $e_{\mathbf{p}}$ *to* $e_{\mathbf{g}}$*, maps points exterior to* $\mathbb{D}_{\mathbf{p}}$ *to points inside* $\mathbb{D}_{\mathbf{g}}$*, and is* C^k *continuous and invertible.*

We will assemble surfaces from patches. A *patch* is the image of a subdomain under a C^k map into $\mathbb{R}^3$. The prefix 'sub' of subdomain points to the fact that a typical C^k map, say a polynomial map, is well-defined on a larger domain.

Definition 3 *Two regular* C^k *maps* $\mathbf{p} : \mathbb{D}_{\mathbf{p}} \to \mathbb{R}^m$ *and* $\mathbf{g} : \mathbb{D}_{\mathbf{g}} \to \mathbb{R}^m$, join G^k via the C^k reparametrization $\mathbf{r}$ between ($\mathbb{D}_{\mathbf{p}}$,e) and ($\mathbb{D}_{\mathbf{g}}$,$\mathbf{r}(e)$) *if*

$$D^\kappa \mathbf{p}(e) = D^\kappa(\mathbf{g} \circ \mathbf{r})(e), \quad \textit{for } \kappa = 0, \dots, k,$$

where $\circ$ *denotes composition, and* $D^\kappa \mathbf{p}(e)$ *is a univariate map obtained by restricting* $D^\kappa \mathbf{p}$ *to* e*. If* $\mathbf{r}$ *is a rigid transformation then* $\mathbf{p}$ *and* $\mathbf{g}$ *join* parametrically C^k*. If* $\mathbf{r}$ *is the identity then* $\mathbf{g}$ *restricted to* $\mathbb{D}_{\mathbf{g}}$ *and* $\mathbf{p}$ *restricted to* $\mathbb{D}_{\mathbf{p}}$ *form a* C^k map *on* $\mathbb{D}_{\mathbf{p}} \cup \mathbb{D}_{\mathbf{g}}$.

If $m = 3$, $\mathbf{p}$ and $\mathbf{g}$ will be called *geometry maps* to emphasize that they determine the local shape and curvature of the surface. The patch $\mathbf{g} \circ \mathbf{r}(\square) \subset \mathbb{R}^3$ may be viewed as the result of 'trimming' a standard domain $\triangle$ of $\mathbf{g}$ (dashed triangle in Figure 2) to the

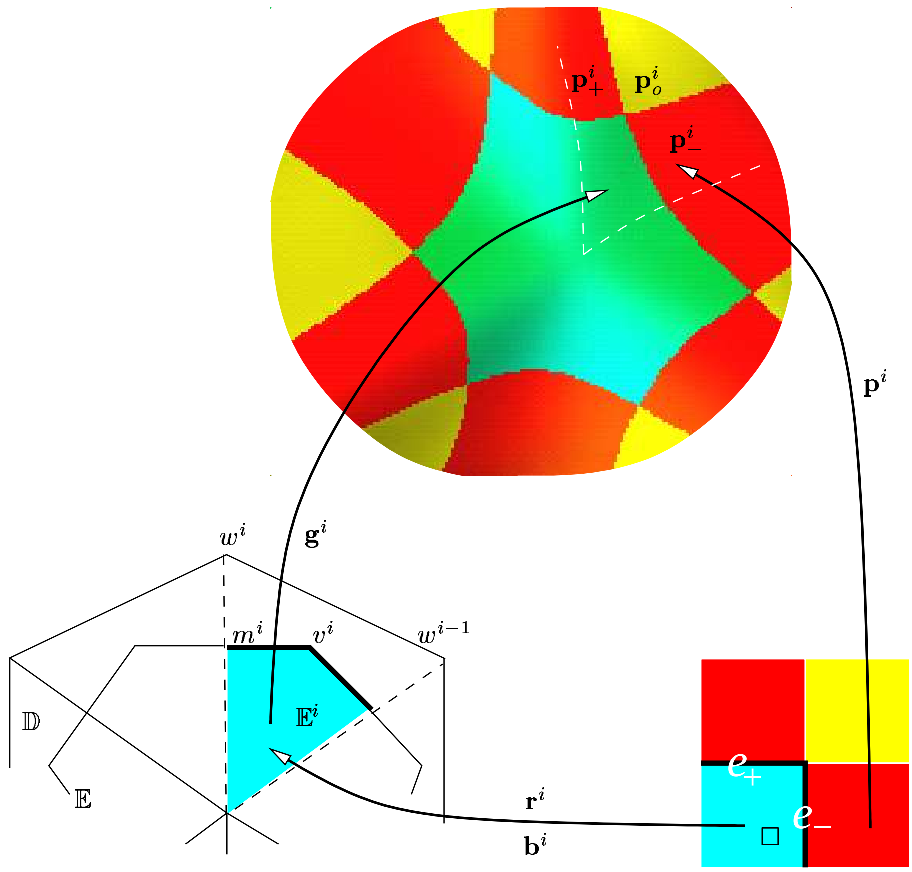

Figure 3: The maps of Theorem 1: The reparametrizations $\mathbf{r}^i$ and $\mathbf{b}^i$ both map the unit square $\square$ to $\mathbb{E}^i \subset \mathbb{R}^2$ and the edges e_- and e_+ to halfedges $[m^{i-1}..v^i]$, $[v^i..m^i]$ of the regular $\mathtt{n}$-disk $\mathbb{E}$; $\mathbb{E}$ is inscribed in a larger, $\frac{\pi}{\mathtt{n}}$-rotated $\mathtt{n}$-disk $\mathbb{D}$ with vertices w^i; $\mathbb{D}$ is the union of the domain triangles $\mathbb{D}^i$ of the maps $\mathbf{g}^i : \mathbb{D}^i \to \mathbb{R}^3$. $\mathbf{p}^i$ maps the closure of $2\square - \square$ to $\mathbb{R}^3$; it consists of the polynomial pieces $\mathbf{p}_+^i$, $\mathbf{p}_o^i$ and $\mathbf{p}_-^i$.

quadrilateral $\mathbb{D}_{\mathbf{g}} := \mathbf{r}(\square)$ where $\square$ is a subdomain adjacent to $\mathbb{D}_{\mathbf{p}}$, the square on the right in Figure 2.

The key to minimizing the degree is to choose a reparametrization that is of low degree along a domain edge e. Then the composition $\mathbf{g} \circ \mathbf{r}(e)$ is of low degree: specifically, if $d^{\kappa}_{\mathbf{r},e} = \kappa + 1$ for $\kappa = 0, 1, 2$ and $\mathbf{g}$ is of total degree d then a simple calculation (cf. the table in the proof of Theorem 1) establishes $d^{\kappa}_{\mathbf{g}\circ\mathbf{r},e} = d + \kappa$.

We apply this observation to filling an $\mathtt{n}-$sided hole using the following (sub)domains. Here and later, $i \in \{1, 2, \dots, \mathtt{n}\}$, $\mathtt{n} \geq 3$, and indices are cyclic, i.e. $i + 1$ maps to 1 for $i = \mathtt{n}$ and $i - 1$ maps to $\mathtt{n}$ for $i = 1$.

Definition 4

$\square$ *denotes the unit square* $[0..1] \times [0..1]$

with upper and right edges $e_+ := [0..1] \times \{1\}$ *and* $e_- := \{1\} \times [0..1]$*;*

$\daleth$ *is the closure of* $2\square - \square$.

$\mathbb{D}$ *is the* $\mathtt{n}$*-disk with vertices* $w^i := \big(\cos(\frac{2\pi i}{\mathtt{n}} + \frac{\pi}{\mathtt{n}}), \sin(\frac{2\pi i}{\mathtt{n}} + \frac{\pi}{\mathtt{n}})\big)$*;*

$\mathbb{D}^i$ *is the triangle with vertices* $0, w^{i-1}, w^i$*;*

$\mathbb{E} \subset \mathbb{D}$ *is an* $\mathtt{n}$*-disk with vertices* $v^i := \sigma\big(\cos(\frac{2\pi i}{\mathtt{n}}), \sin(\frac{2\pi i}{\mathtt{n}})\big)$, $0 < \sigma \leq \cos(\frac{\pi}{n})$,

edge-midpoints $m^i := (v^{i+1} + v^i)/2$*, boundary* $\partial\mathbb{E}$ *, and*

quadrilateral sectors $\mathbb{E}^i$ *with vertices* $0, m^{i-1}, v^i, m^i$.

2 Smooth Joining of Trimmed Patches

The following central theorem establishes the connection between four piecewise polynomial maps. For each map, we list (1) the number of pieces, their domain and range, (2) properties of the image, (3) smoothness and regularity, and (4) degree. Figure 3 sketches the maps. Note that $\mathbf{r}$, $\mathbf{g}$ and $\mathbf{p}$ in Figure 2 correspond to $\mathbf{r}^i$, $\mathbf{g}^i$ and $\mathbf{p}^i$ below.

Theorem 1

Let $\mathbf{r}$ *be a* reparametrization *with the following properties (cf. Figure 4)*

(r_1) $\mathbf{r}$ *consists of* $\mathtt{n}$ *polynomial pieces* $\mathbf{r}^i : \square \to \mathbb{R}^2$.

(r_2) $\mathbb{E}^i = \mathbf{r}^i(\square)$, $\mathbf{r}^i(e_+) = [v^i..m^i]$, $\mathbf{r}^i(e_-) = [v^i..m^{i-1}]$.

(r_3) *For* $\kappa = 0, 1, 2$, $D^{\kappa}\mathbf{r}^i(\{0\} \times [0..1]) = D^{\kappa}\mathbf{r}^{i+1}([0..1] \times \{0\})$*;*

$\det\big(D\mathbf{r}^i(e_- \cup e_+)\big) > 0$.

(r_4) *For* $\kappa = 0, 1, 2$, $d^{\kappa}_{\mathbf{r},e_+} = d^{\kappa}_{\mathbf{r},e_-} = \kappa + 1$.

Let $\mathbf{g}$ *be a* geometry map *with the following properties (cf. Figure 6).*

(g_1) $\mathbf{g}$ *consists of* $\mathtt{n}$ *polynomial pieces* $\mathbf{g}^i : \mathbb{D}^i \to \mathbb{R}^3$.

(g_2) *For all* i, $\mathbf{g}^i(0) = P$ *and* $\mathbf{g}(\mathbb{D}) = \bigcup_i \mathbf{g}^i(\mathbb{D}^i)$ *is a regular surface neighborhood of* P.

(g_3) *Each of the three spatial components of* $\mathbf{g}^i$ *and* $\mathbf{g}^{i+1}$ *form a* C^2 *function on* $\mathbb{D}^i \cup \mathbb{D}^{i+1}$.

(g_4) *The total degree of* $\mathbf{g}^i$ *is* d.

Then there exists a piecewise polynomial map $\mathbf{p} : \{1, \dots, \mathtt{n}\} \times \mathbb{R}^2 \to \mathbb{R}^3$ *such that*

(p_1) $\mathbf{p}$ *consists of* $\mathtt{n}$ *piecewise polynomials* $\mathbf{p}^i : \daleth \to \mathbb{R}^3$.

(p_2) $\bigcup_i \mathbf{p}^i(\daleth)$ *forms a regular surface extension of* $\mathbf{g}(\mathbb{D})$.

(p_3) $\mathbf{p}^{i+1}$ *and* $\mathbf{p}^i$ *join parametrically* C^2

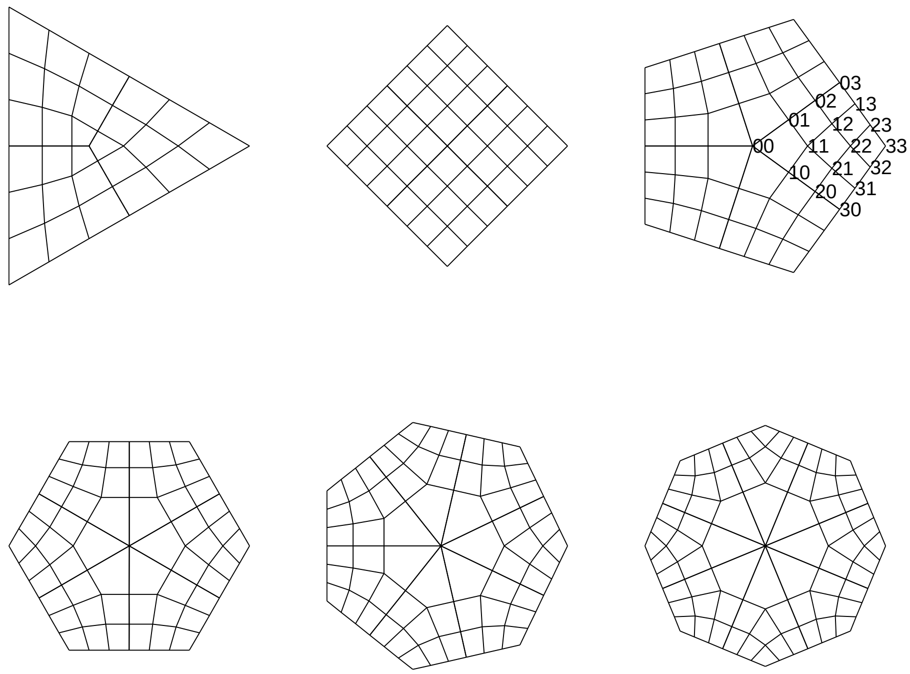

Figure 4: For $\mathtt{n} = 3, \ldots, 8$, the Bézier control points of $\mathtt{n}$ bicubic maps $\mathbf{r}^i : \Box \to \mathbb{R}^2$, $i = 1, \ldots, \mathtt{n}$, that join parametrically C^2 at the midpoints of the $\mathtt{n}$-gon boundary (labeled 03, resp. 30).

across $\mathbf{p}^i(\{0\} \times [1..2]) = \mathbf{p}^{i+1}([1..2] \times \{0\})$.
$\mathbf{p}^i$ *and* $\mathbf{g}^i \circ \mathbf{r}^i$ *form a* C^2 *map on* $2\Box$.
($\mathtt{p}_4$) *For* $\kappa = 0, 1, 2$, $d^\kappa_{\mathbf{p}^i, e_+} = d^\kappa_{\mathbf{p}^i, e_-} = d + \kappa$.

Moreover, there exists a map $\mathbf{b}$ *consisting of* $\mathtt{n}$ *polynomial pieces* $\mathbf{b}^i$ *such that*
($\mathtt{b}_1$) $\mathbf{b}^i : \Box \to \mathbb{E}^i, (s,t) \mapsto m^{i-1}\,(1-s)t + v^i\,st + m^i\,s(1-t)$.
($\mathtt{b}_2$) $\mathbf{g}(\mathbb{E}) = \bigcup_i \mathbf{g}^i \circ \mathbf{b}^i(\Box)$.
($\mathtt{b}_3$) $\mathbf{g}^i \circ \mathbf{b}^i$ *joins* G^2 *with* $\mathbf{p}^i$ *across* $\mathbf{g}^i \circ \mathbf{b}^i(e_- \cup e_+) = \mathbf{p}^i(e_- \cup e_+)$ *and with* $\mathbf{g}^{i+1} \circ \mathbf{b}^{i+1}$ *across* $\mathbf{g}^i \circ \mathbf{b}^i(\{0\} \times [0..1]) = \mathbf{g}^{i+1} \circ \mathbf{b}^{i+1}([0..1] \times \{0\})$.
($\mathtt{b}_4$) $\mathbf{g}^i \circ \mathbf{b}^i$ *is of degree* (d, d).

Examples of maps $\mathbf{r}$ for various $\mathtt{n}$ are shown in Figure 4. Examples of geometry maps $\mathbf{g}$ are shown in Figure 7, *right, bottom*. Figure 9 shows Hermite extensions $\mathbf{p}$ (*dark grey*) and vertex neighborhoods $\mathbf{g}(\mathbb{E})$ (*medium grey*); for $d = 3$, $\mathtt{b}_2$ implies that $\mathbf{g}(\mathbb{E})$ can be parametrized by $\mathtt{n}$ bicubic tensor-product polynomials $\mathbf{g}^i \circ \mathbf{b}^i : \Box \to \mathbb{R}^3$.

Proof For $\kappa = 0, 1, 2$, we define $D^\kappa \mathbf{p}^i := D^\kappa(\mathbf{g}^i \circ \mathbf{r}^i)$ on $e_+ \cup e_-$. Since $\mathbf{g}$ is C^2 and since $\mathbf{r}^i$ and $\mathbf{r}^{i+1}$ join parametrically C^2, the Taylor expansions of $\mathbf{p}^{i+1}$ and $\mathbf{p}^i$ join parametrically C^2 where they are determined by $\mathbf{g}^i \circ \mathbf{r}^i$: $D^\kappa \mathbf{p}^{i+1}(1,0) = D^\kappa \mathbf{p}^i(0,1)$

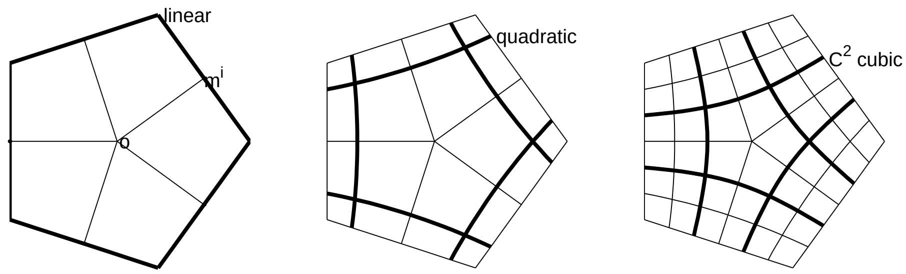

Figure 5: The piecewise bicubic map $\mathbf{r}$ is defined by rings of linear, quadratic and cubic curves.

for $\kappa = 0, 1, 2$; since $\mathbf{p}$ is not further constrained, its expansion can be completed to a C^2 spline satisfying $\mathtt{p}_1$, $\mathtt{p}_2$, $\mathtt{p}_3$ (see, for example, the construction in Section 3). Dropping the superscripts, subscripts and reference to the edge and applying the chain rule to $\mathbf{g} \circ \mathbf{r}$, we obtain the table below. The left column lists the expansion of the composition, i.e. Fáa di Bruno's Law for $\kappa = 0, 1, 2$, the center column lists the maximal degree of the expansion restricted to the edge e given that $d^0_{\mathbf{r}} = 1$, and the last column lists the degree due to $d^\kappa_{\mathbf{r}} = \kappa + 1$ and proves $\mathtt{p}_4$.

derivative	degree ($d^0_{\mathbf{r}} = 1$)	$=$
$\mathbf{g} \circ \mathbf{r}$	$d^0_{\mathbf{g}}$	d
$(D\mathbf{g}) \circ \mathbf{r}\langle D_{\mathtt{e}^\perp}\mathbf{r}\rangle$	$d^1_{\mathbf{g}} + d^1_{\mathbf{r}}$	$d+1$
$(D^2\mathbf{g}) \circ \mathbf{r}\langle D_{\mathtt{e}^\perp}\mathbf{r}, D_{\mathtt{e}^\perp}\mathbf{r}\rangle + (D\mathbf{g}) \circ \mathbf{r}\langle D^2_{\mathtt{e}^\perp}\mathbf{r}\rangle$	$\max\{d^2_{\mathbf{g}} + 2d^1_{\mathbf{r}}, d^1_{\mathbf{g}} + d^2_{\mathbf{r}}\}$	$d+2$

$\mathtt{b}_2$ follows from $\mathbf{b}^i(\square) = \mathbb{E}^i = \mathbf{r}^i(\square)$ and, evidently, $\mathbf{g}^i \circ \mathbf{b}^i$ is of degree (d, d) and joins G^2 with $\mathbf{p}^i$ via $(\mathbf{b}^i)^{-1} \circ \mathbf{r}^i$. Since $\mathbf{b}^i$ only trims the domain of the smoothly connected spline pieces $\mathbf{g}^i$ to $\mathbb{E}^i$, $\mathtt{b}_3$ holds. |||

The identification of edges via the reparametrizations $\mathbf{r}$ define the connecting relations of a topological space that can serve as global domain of the spline construction in the sense of [GH95, Rei99, NG00].

We now construct maps $\mathbf{g}$ and $\mathbf{r}$ that satisfy the assumptions of Theorem 1. A suitable **reparametrization** $\mathbf{r} : \{1, \dots, \mathtt{n}\} \times \square \to \mathbb{R}^2$ consists of $\mathtt{n}$ bicubic polynomial pieces with Bézier control points r^i_{jk}, $i = 1, \dots, \mathtt{n}$ where $r^i_{jk} = \left[\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}\right]^i r^{\mathtt{n}}_{jk}$, $c := \cos(2\pi/\mathtt{n})$, $s := \sin(2\pi/\mathtt{n})$. Explicitly, with the upper left entry corresponding to $r^{\mathtt{n}}_{00}$ and the lower right to $r^{\mathtt{n}}_{33}$, and $\sigma_{\text{default}} = \cos(\pi/\mathtt{n})$, the x components of $r^{\mathtt{n}}$ are

$$\frac{\sigma}{72}\begin{bmatrix} 0 & (1+c)\,(15+11\,c) & 6\;(1+c)\,(4+c) & 36+36\,c \\ (1+c)\,(15+11\,c) & 30+22\,c & 42+26\,c & 48+24\,c \\ 6\;(1+c)\,(4+c) & 42+26\,c & 48+16\,c & 60+12\,c \\ 36+36\,c & 48+24\,c & 60+12\,c & 72 \end{bmatrix}$$

and the y components of $r^{\mathtt{n}}$ are

$$\frac{\sigma}{72}\sin(2\pi/\mathtt{n})\begin{bmatrix} 0 & 15+11\,c & 24+6\,c & 36 \\ -15-11\,c & 0 & 12+2\,c & 24 \\ -24-6\,c & -12-2\,c & 0 & 12 \\ -36 & -24 & -12 & 0 \end{bmatrix}.$$

The maps are best understood by their construction which consists of setting r^i_{00} to be the origin and the remaining 15 control points of each degree $(3,3)$ piece to be the control points of symmetric families of curve segments in Bézier form.

a. The outermost curves are the linear segments of $\partial\mathbb{E}$. Expressing these in Bézier form of degree 3 and subdividing at the midpoint creates 7 control points (Figure 5, *right*) per edge, i.e. the control points r^i_{j3} and r^i_{3j} for $j=0,1,2,3$.

b. The 7 control points of the second layer from each outer boundary represent a quadratic curve segment (Figure 5, *middle*), raised to degree 3 and subdivided at the midpoint. The first and the last Bézier control point of the ith quadratic segment are r^i_{32} and r^{i+1}_{23}. Subdividing at the midpoint and placing the middle control point of the quadratic on the diagonal from r^i_{00} to r^i_{33} uniquely determines the quadratic curve pieces. Degree-raising yields the control points r^i_{j2} and r^i_{2j}, $j=0,1,2$.

c. The third layer of coefficients are the control points of a piecewise cubic curve (Figure 5, *right*). Two control points of each cubic are r^i_{31}, r^i_{21} respectively r^{i+1}_{13} and r^{i+1}_{12}. Two additional control points are pinned down by the C^2 join, and the shared final degree of freedom is used to place r^i_{11} on the diagonal. This defines the remaining control points r^i_{j1} and r^i_{1j}, $j=0,1$.

Lemma 2.1 *The map* $\mathbf{r}:\{1,\dots,\mathtt{n}\}\times\square\to\mathbb{R}^2$ *defined above has properties* $\mathtt{r}_1$–$\mathtt{r}_4$ *of Theorem 1.*

Proof Properties $\mathtt{r}_1$ and $\mathtt{r}_2$ are explicit in the construction. Since $D^\kappa\mathbf{r}^i(e_+)$ and $D^\kappa\mathbf{r}^{i+1}(e_-)$ are two pieces of the same polynomial for $\kappa=0,1$ and two pieces of a C^2 spline for $\kappa=2$, the pieces join parametrically C^2 at m^i (via the reflection that maps e_- to e_+). Evidently, they are of degree $\kappa+1$. Regularity and, by the inverse function theorem, local invertibility of $\mathbf{r}$ in a neighborhood of $\partial\mathbb{E}$ are verified by computing the Jacobian. Restricted to the boundary, the Jacobian is a positive multiple of the univariate polynomial with Bézier coefficients $20-10c, 20-10c, 20-11c, 20-13c, 20-16c, 20-20c$. |||

It is instructive to compare $\mathbf{r}$ with three other bicubic reparametrizations that play a central role in higher-order constructions. The reparametrization in [Pra97] is defined by the subdominant eigenfunctions of Catmull-Clark subdivision. [Rei98] defines a singular parametrization where r^i_{jk} for $j,k<3$ coalesce while the remaining control points form a curved $\mathtt{n}$-gon with right angles at the vertices. The construction in

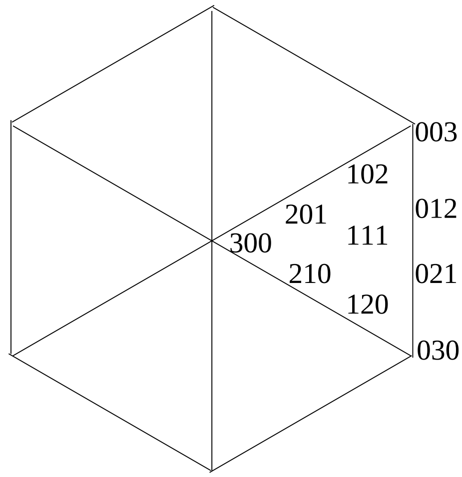

Figure 6: Coefficient indices of a cubic spline piece on $\mathbb{D}$.

[NG00] uses rotated copies of a map of degree $(3, 1)$, derived as the parallel sweep of a cubic curve. All three reparametrizations have $d^0_{\mathbf{r},e} = 3$ and therefore the composition $\mathbf{g} \circ \mathbf{r}$ has degree $3d$ rather than d. There is, however, a trade-off. The cubic boundary curves of the three reparametrizations allow for curved boundaries with control over the angles at the vertices of the curved $\mathtt{n}$-gon. The linear boundary curves of $\partial\mathbb{E}$, by contrast, intersect with an angle that converges to 0 as $\mathtt{n}$ goes to infinity. While this near-singularity for large $\mathtt{n}$ is in principle no problem (see the singular construction of [Rei98]), parametrization-dependent applications must proced with care.

We choose the **geometry map** as a C^2 spline with pieces $\mathbf{g}^i$ of total degree d over a triangulated unit $\mathtt{n}$-disk $\mathbb{D}$ as shown in Figure 6. For $d = 1$, such a C^2 spline is a linear polynomial and for $d = 2$, it is a quadratic polynomial. For $d \geq 3$, counting shows that the spline has at least $6 + \mathtt{n}\big(\binom{d+2}{2} - 3d\big)$ degrees of freedom in each spatial component. For the important case $d = 3$, we analyze the distribution and exact number of free coefficients in more detail. Figure 6 shows the Bézier index triples $\alpha = \alpha_0\alpha_1\alpha_2$ of one cubic spline piece $\mathbf{g}^i$, $i \in \{1, \ldots, \mathtt{n}\}$

$$\mathbf{g}^i := \sum_{\alpha_0+\alpha_1+\alpha_2=3} g^i_\alpha B_\alpha, \quad B_\alpha(u,v) := \frac{3!}{\alpha_0!\alpha_1!\alpha_2!}(1-u-v)^{\alpha_0} u^{\alpha_1} v^{\alpha_2}.$$

The B_α are the cubic Bézier basis functions. We prove that such a spline has at least one degree of freedom for each boundary curve.

Lemma 2.2 *A C^2 spline consisting of $\mathtt{n}$ polynomial pieces $\mathbf{g}^i : \mathbb{D}^i \to \mathbb{R}$ of total degree 3 (cf. Figure 6) has at least $\mathtt{n} + 6$ free coefficients. These may be chosen as*

$$g^1_{300}, g^1_{210}, g^1_{201}, g^1_{120}, g^1_{111}, g^1_{102},$$

that determine the quadratic Taylor expansion at 0*, and*

$$g^i_{003}, i = 1, \dots, \mathtt{n},$$

the outermost Bézier coefficient of each boundary curve. If $\mathtt{n} = 3$ *or* $\mathtt{n} = 6$*, then one additional coefficient, e.g.* g^1_{021}*, can be prescribed. If* $\mathtt{n} = 4$*, then two additional coefficients, e.g.* g^1_{021} *and* g^1_{012}*, can be prescribed.*

Proof The parametric C^2 transitions and $g^1_{\alpha_0\alpha_1\alpha_2}$ for $\alpha_0 > 0$ determine the coefficients with $\alpha_0 > 0$. Since three consecutive vertices of the $\mathtt{n}$-gon are related by $\left[\begin{smallmatrix} c \\ s \end{smallmatrix}\right] + \left[\begin{smallmatrix} c \\ -s \end{smallmatrix}\right] = 2c\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ where $c := \cos(2\pi/\mathtt{n}), s := \sin(2\pi/\mathtt{n})$, the remaining C^1 conditions for $i = 1, \dots, \mathtt{n}$ are

$$g^i_{012} - g^i_{102} + g^{i+1}_{021} - g^i_{102} = 2c\big(g^i_{003} - g^i_{102}\big).$$

Solving for g^i_{012} and substituting into the C^2 constraints yields the circulant system

$$g^{i+2}_{021} + 4cg^{i+1}_{021} + g^i_{021} = G_i$$

where G_i is a function of g^j_{003} and g^j_α with $\alpha_0 > 0$, $j \in \{i-1, i, i+1\}$. The corresponding tridiagonal circulant matrix is of full rank unless $\cos(2\pi k/\mathtt{n}) = -2c$ for some $k \in \{0, \dots, \mathtt{n}-1\}$. For $\mathtt{n} > 6$ we have $2c > 1$ and the lemma holds. The remaining cases are verified directly. |||

3 Construction of a spline surface: an algorithm

The previous section creates an algebraic foundation for building flexible C^2 surfaces of low degree. As an example of how Theorem 1 may be applied, this section gives a recipe for smoothing out a polyhedral mesh with quadrilateral facets (see Figure 7, *top*). The recipe is as follows:

1. Determine the maps $\mathbf{g}$ and define the vertex neighborhoods by $\mathbf{g} \circ \mathbf{b}$ (*medium grey* in Figure 9).

2. Connect any two adjacent neighborhoods by splines of degree $(3, d+2)$ (*dark grey*) that Hermite-interpolate the boundary data.

3. Repeat the interpolation process to cover the quadrilateral faces of the original shape with tensor-product splines of degree $(3, 3)$ (*light grey*).

The recipe has two flaws that will not be fixed in this paper. As $\mathtt{n}$ becomes large, $\mathbf{r}^i$ becomes singular at $(1, 1)$ since the $\mathtt{n}$-gons converge to a circle. There is no proof of a local convex hull property, a flaw shared by almost all polynomial schemes listed in the introduction. We note that other schemes, say similar to [MP96], can be based on Theorem 1.

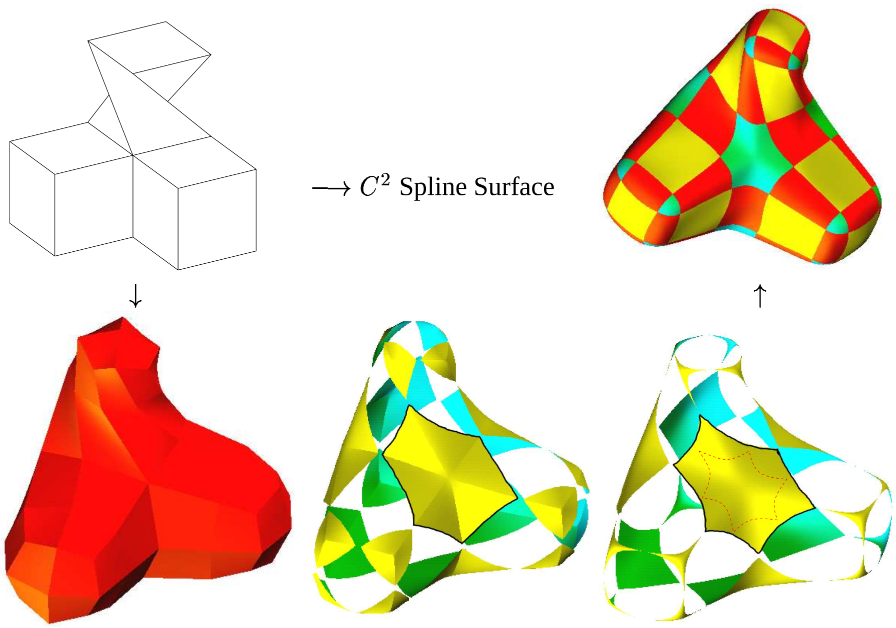

Figure 7: Construction of a spline surface. (*left*) The input control polyhedron and one Catmull-Clark refinement step. (*middle, bottom*) Star-shaped clusters of triangular, total degree 3 proposal patches $\bar{\mathbf{g}}$ obtained from the Catmull-Clark mesh. Backfacing clusters are visible since the clusters only cover part of the refined Catmull-Clark mesh. The central hexagonal cluster is outlined for clearity. (*right, bottom*) The proposal patches are converted to nearby C^2 patch complex $\mathbf{g}(\mathbb{D})$ using singular value decomposition. The vertex neighborhoods $\mathbf{g}(\mathbb{E})$ are cut out of these clusters (dashed lines) and then connected by interpolating splines first across edges and then across faces (*right,top*).

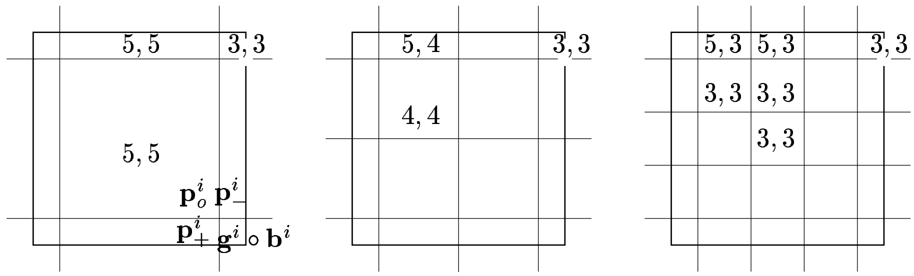

Figure 8: Degree and patch layout choices for covering one input quadrilateral facet.

3.1 Geometry of the vertex neighborhood

The actual geometry of the triangular patches may be determined by minimizing the deviation from a *proposed* set of control points $\bar{g}^i_\alpha$ of our choice and enforcing smoothness across the radial edges $[0..w^i]$ of $\mathbb{D}$:

$$\min_{g^i_\alpha} \sum_{i=1}^{n} \sum_{\alpha_0+\alpha_1+\alpha_2=3} \|g^i_\alpha - \bar{g}^i_\alpha\|^2$$
$$D^\kappa \mathbf{g}^i = D^\kappa \mathbf{g}^{i+1} \text{ on } [0..w^i], \text{ for } \quad i = 1, \ldots, \mathtt{n}, \quad \kappa = 0, 1, 2.$$

Since the constraints are linear in the coefficients, the quadratic minimization problem reduces to finding a least solution in the variables $h^i_\alpha := g^i_\alpha - \bar{g}^i_\alpha$ by singular value decomposition. To obtain the coefficients $\bar{g}^i_\alpha$ for $\alpha_0 < 3$ we apply one Catmull-Clark refinement step (Figure 7, *left*). Let $c^i_{\alpha_1\alpha_2}$ be the Bernstein-Bézier coefficients of $\mathtt{n}$ tensor-product patches meeting at a point $P = c^i_{00}$, $i = 1, \ldots, \mathtt{n}$ on the Catmull-Clark limit surface that corresponds to an original mesh node. Then we choose $\bar{g}^i_\alpha = c^i_{\alpha_1\alpha_2}$ for $\alpha_1 + \alpha_2 \leq 3$.

As shown in Figure 7, the surface segments $\mathbf{g}^i$ reach halfways across to the neighbor point of the input control net. Setting σ so that $\mathbf{r}^i(\square) \subset \mathbb{D}$ cuts out four-sided surface pieces $\mathbf{g}^i \circ \mathbf{b}^i(\square) = \mathbf{g}^i \circ \mathbf{r}^i(\square)$ that combine to form $\mathtt{n}$-sided vertex neighborhoods (*medium grey* in Figure 9) creates space for the first three layers of Bézier control points of the Hermite extension $\mathbf{p} = \mathbf{g} \circ \mathbf{r}$ (*dark grey* in Figure 9). The resulting vertex neighborhoods can each independently be affinely transformed. For example they can be scaled by local parameters $\mathtt{g}_{\mathtt{sc}} \in [0..1]$ to distribute curvature, i.e. sharpen or smooth out the edge neighborhoods as in [Pet94].

3.2 Hermite interpolation of vertex neighborhoods

For each vertex neighborhood and $d = 3$, we now have (*medium grey* in Figure 9) a parametrization by patches $\mathbf{g}^i \circ \mathbf{b}^i$ of degree $(3, 3)$ and the Taylor expansion up to order two of a complex p that parametrically C^2 extends $\mathbf{g} \circ \mathbf{r}$. Connecting the vertex neighborhoods by interpolating the Taylor expansions to form edge patches amounts to constructing a univariate C^2 spline for each parameter line of $\mathbf{p}^i$ (*dark grey* in Figure 9)

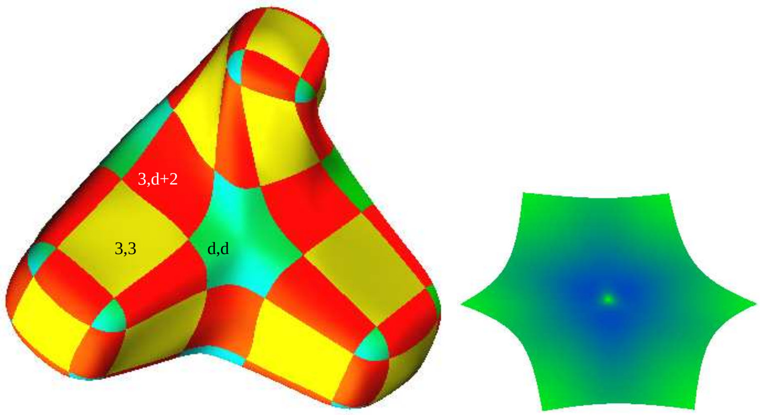

Figure 9: (*Left:*) The vertex neighborhoods (*medium grey*) consist of patches of degree $(3,3)$. Faces (*light grey*) are C^2 splines of degree $(3,3)$, while each edge (*dark grey*) is covered by a C^2 spline of degree $(3,5)$. Edge and face neighborhoods join parametrically C^2 while the vertex neighborhoods join G^2 with the rest of the surface. (*Right:*) The Gauss curvature of the higher-order (monkey) saddle on the *left* with its three dips and three raises. The Gausss curvature is zero (*green, medium grey*) at the central point and negative (*blue, darker grey*) in the surrounding region.

and interpolating the given Taylor expansion at each end. Six such interpolants suffice to define the degree $(3,5)$ edge patches. The face patches of degree $(3,3)$ (*light grey* in Figure 9) are constructed by tensoring the approach. Layout and degree choices for the completion of the surface are sketched in Figure 8.

4 Summary and higher-order continuity

The reduction of the maximal degree of geometrically flexible C^2 free-form surfaces from degree $(9,9)$ to degree $(5,3)$ makes use of the following ideas.

- The map defining the geometry in the neighborhood of an extraordinary point is defined by a *spline* and the x, y and z components of the spline are of *total degree*, here 3.
- The reparametrization changes *(trims) the domain* of each spline piece from a triangle to a rectangular domain.
- The reparametrization is only relevant in the neighborhood of the edges of an $\mathtt{n}$-gon (to define the connection between the vertex neighborhood and its tensor-product surrounding); the vertex neighborhood has two different parametrizations that yield the same point set.

To increase the smoothness of the surface contruction, we only need to find a reparametrization $\mathbf{r}$, of low degree, that is parametrically C^k at all preimages of $\partial\mathbb{E}$. Fortunately, the construction of $\mathbf{r}$ in terms of polynomial curves generalizes: $D^\kappa \mathbf{r}^{i+1}(e_-)$ and $D^\kappa \mathbf{r}^i(e_+)$ must match κ control points so that $\mathbf{r}^i$ is $C^{k,k}$ at (1,1). These 2κ plus the symmetry requirement are exactly met by a polynomial of degree 2κ. According to the Appendix, this yields a composition $\mathbf{g}^i \circ \mathbf{r}^i$ with $d^k_{\mathbf{p}^i,e_-} = d^k_{\mathbf{p}^i,e_+} = 2k+d-1$. The degree can be reduced to $2k+d-2$ by using a C^k spline in the last step of the contruction of $\mathbf{r}$. Since $k+1$ univariate, C^k-connected polynomial pieces of degre $k+1$ can interpolate kth order Hermite data at each end, this results in a construction of degree $(k+1, 2k+d-2)$. If, for $k \geq 2$, one chooses $d = k+1$ to obtain $\mathtt{n}$ degrees of freedom this yields a C^k surface parametrized by tensor-product splines of degree $(k+1, 3k-1)$ as compared to degree $((k+1)^2, (k+1)^2)$ for [Pra97, Rei98].

Acknowledgements: The notation owes much to careful reading by Carl de Boor, Kestas Karciauskas, Hartmut Prautzsch and Ulrich Reif.

References

[BR97] H. Bohl and U. Reif. Degenerate Bézier patches with continuous curvature. *Comput. Aided Geom. Design*, 14(8):749–761, 1997.

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10:350–355, September 1978.

[CS96] G. M. Constantine and T. H. Savits. A multivariate Faà di Bruno formula with applications. *Trans. Amer. Math. Soc.*, 348(2):503–520, 1996.

[DL89] T. DeRose and C. Loop. The S-patch: a new multisided patch scheme. *ACM Trans. on Graphics*, 8(3):204–234, 1989.

[Du88] Wen-Hui Du. *Etude sur la représentation de surfaces complexes: application à la réconstruction de surfaces échantillonnées*. PhD thesis, Ecole Nationale Supeŕieure des Télécommunications, Paris, France, 1988.

[GH89] John A. Gregory and Jörg M. Hahn. A C^2 polygonal surface patch. *Comput. Aided Geom. Design*, 6(1):69–75, 1989.

[GH95] C. M. Grimm and J. F. Hughes. Modeling surfaces of arbitrary topology using manifolds. *Computer Graphics*, 29(Annual Conference Series):359–368, 1995.

[GZ94] John A. Gregory and Jianwei Zhou. Filling polygonal holes with bicubic patches. *Computer Aided Geometric Design*, 11(4):391–410, 1994.

[GZ99] John A. Gregory and Jianwei Zhou. Irregular C^2 surface construction using bi-polynomial rectangular patches. *Comput. Aided Geom. Design*, 16(5):423–435, 1999.

[Hah89] Jörg Hahn. Filling polygonal holes with rectangular patches. In *Theory and practice of geometric modeling (Blaubeuren, 1988)*, pages 81–91. Springer, Berlin, 1989.

[Her96] T. Hermann. G^2 interpolation of free-form curve networks by biquintic Gregory patches. *Comput. Aided Geom. Design*, 13:873–893, 1996.

[Knu69] D. E. Knuth. *The art of computer programming*, volume 1, 2, 3. Addison Wesley, 1969.

[MP96] Thomas Müller and Hartmut Prautzsch. *A free form spline software package*. Universität Karlsruhe, 1996.

[NG00] J. Cotrina Navau and N. Pla Garcia. Modeling surfaces from meshes of arbitrary topology. *Comput. Aided Geom. Design*, 17(7):643–671, 2000.

[Pet94] J. Peters. Surfaces of arbitrary topology constructed from biquadratics and bicubics. In Nickolas S. Sapidis, editor, *Designing fair curves and surfaces: shape quality in geometric modeling and computer-aided design*, pages 277–293. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1994.

[Pet96] J. Peters. Curvature continuous spline surfaces over irregular meshes. *Computer-Aided Geometric Design*, 13(2):101–131, Feb 1996.

[Pet00] J. Peters. Patching Catmull-Clark meshes. *Siggraph 2000*, pages 255–258, 2000.

[Pra97] Hartmut Prautzsch. Freeform splines. *Comput. Aided Geom. Design*, 14(3):201–206, 1997.

[PU00] H. Prautzsch and G. Umlauf. Triangular G^2 splines. In L.L. Schumaker P.J. Laurent, A. LeMéhauté, editor, *Curve and Surface Design*, pages 335–342. Vanderbilt University Press, 2000.

[Rei95] Ulrich Reif. Biquadratic G-spline surfaces. *Computer Aided Geometric Design*, 12(2):193–205, 1995.

[Rei98] U. Reif. TURBS—topologically unrestricted rational B-splines. *Constr. Approx.*, 14(1):57–77, 1998.

[Rei99] Ulrich Reif. *Analyse und Konstruktion von Subdivisionsalgorithmen für Freiformflächen beliebiger Topologie*. Shaker Verlag, Aachen, 1999.

[Sab91] Malcolm Sabin. Cubic recursive division with bounded curvature. In L.L. Schumaker P.J. Laurent, A. LeMéhauté, editor, *Curves and Surfaces*, pages 411–414. Academic Press, 1991.

[Ye97] Xiuzi Ye. Curvature continuous interpolation of curve meshes. *Computer Aided Geometric Design*, 14(2):169–190, 1997.

5 Appendix: Fáa di Bruno's chain rule Law

Faá di Bruno's Law combines the chain rule and the product rule. In one variable where $D^\kappa \mathbf{g}$ denotes the κth derivative of $\mathbf{g}$

$$D^\kappa(\mathbf{g}\circ\mathbf{r}) = \sum_{j=0}^{\kappa}\sum_{K(j)} c_{K(j)}\Big((D^j\mathbf{g})\circ\mathbf{r}\Big)\cdot(D^1\mathbf{r})^{k_1}\cdot\ldots\cdot(D^\kappa\mathbf{r})^{k_\kappa}$$

$$K(j) := \{k_i \geq 0, i = 1,\ldots,\kappa,\ \sum_{i=1}^{\kappa} k_i = j,\ \sum_{i=1}^{\kappa} ik_i = \kappa\},\ c_{K(j)} := \frac{\kappa!}{k_1!(1!)^{k_1}\cdots k_\kappa!(\kappa!)^{k_\kappa}}.$$

In the multivariate case (see e.g. [CS96]), $\langle\ldots\rangle$ encloses the arguments of the multilinear form $(D^j\mathbf{g})\circ\mathbf{r}$ and the multiplicity of the arguments is expressed by superscripts:

$$D^\kappa(\mathbf{g}\circ\mathbf{r}) = \sum_{j=0}^{\kappa}\sum_{K(j)} c_{K(j)}\Big((D^j\mathbf{g})\circ\mathbf{r}\Big)\langle(D^1\mathbf{r})^{k_1},\ldots,(D^\kappa\mathbf{r})^{k_\kappa}\rangle.$$

The derivatives in the direction $\mathsf{e}^\perp$, evaluated along an edge e are

$$(D^\kappa_{\mathsf{e}^\perp}(\mathbf{g}\circ\mathbf{r}))(e) = \sum_{j=0}^{\kappa}\sum_{K(j)} c_{K(j)}\Big((D^j\mathbf{g})\circ\mathbf{r}(e)\Big)\langle(D^1_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_1},\ldots,(D^\kappa_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_\kappa}\rangle.$$

The left column of the table on page 7 shows the expansion for $\kappa = 0, 1, 2$.
If $\mathbf{g}$ is of total degree d and $d^\kappa_{\mathbf{r},e} :=$ degree $\big(D^\kappa_{\mathsf{e}^\perp}\mathbf{r}(e)\big) = \kappa + 1$ then the degree for each summand for $\kappa > 0$ is

$$\begin{aligned}\text{degree}\ \underbrace{\Big((D^j\mathbf{g})\circ\mathbf{r}(e)\Big)}_{d-j}\langle\underbrace{(D^1_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_1}}_{k_1+k_1},\ldots,\underbrace{(D^\kappa_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_\kappa}}_{k_\kappa+\kappa k_\kappa}\rangle &= d - j + \sum_{i=1}^{\kappa} k_i + \sum_{i=1}^{\kappa} ik_i \\ &= d - j + j + \kappa.\end{aligned}$$

If $\mathbf{g}$ is of total degree d and $d^\kappa_{\mathbf{r},e} = 2\kappa$ then the degree for each summand is

$$\begin{aligned}\text{degree}\ \underbrace{\Big((\partial^j\mathbf{g})\circ\mathbf{r}(e)\Big)}_{d-j}\langle\underbrace{(D^1_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_1}}_{2k_1},\ldots,\underbrace{(D^\kappa_{\mathsf{e}^\perp}\mathbf{r}(e))^{k_\kappa}}_{2\kappa k_\kappa}\rangle &= d - j + 2\sum_{i=1}^{\kappa} ik_i = d - j + 2\kappa \\ &\leq d - 1 + 2\kappa.\end{aligned}$$