Review Questions

1.1 True or false: A problem is ill-conditioned if its solution is highly sensitive to small changes in the problem data.

1.2 True or false: Using higher-precision arithmetic will make an ill-conditioned problem better conditioned.

1.3 True or false: The conditioning of a problem depends on the algorithm used to solve it.

1.4 True or false: A good algorithm will produce an accurate solution regardless of the condition of the problem being solved.

1.5 True or false: The choice of algorithm for solving a problem has no effect on the propagated data error.

1.6 True or false: If two real numbers are exactly representable as floating-point numbers, then the result of a real arithmetic operation on them will also be representable as a floating-point number.

1.7 True or false: Floating-point numbers are distributed uniformly throughout their range.

1.8 True or false: Floating-point addition is associative but not commutative.

1.9 True or false: In a floating-point number system, the underflow level is the smallest positive number that perturbs the number 1 when added to it.

1.10 Explain the distinction between truncation (or discretization) and rounding.

1.11 Explain the distinction between absolute error and relative error.

1.12 Explain the distinction between computational error and propagated data error.

1.13 (*a*) What is meant by the *conditioning* of a problem?

(*b*) Is it affected by the algorithm used to solve the problem?

(*c*) Is it affected by the precision of the arithmetic used to solve the problem?

1.14 If a computational problem has a condition number of 1, is this good or bad? Why?

1.15 When is an approximate solution to a given problem considered to be good according to backward error analysis?

1.16 For a given floating-point number system, describe in words the distribution of machine numbers along the real line.

1.17 In floating-point arithmetic, which is generally more harmful, underflow or overflow? Why?

1.18 In floating-point arithmetic, which of the following operations on two positive floating-point operands can produce an overflow?

(*a*) Addition

(*b*) Subtraction

(*c*) Multiplication

(*d*) Division

1.19 In floating-point arithmetic, which of the following operations on two positive floating-point operands can produce an underflow?

(*a*) Addition

(*b*) Subtraction

(*c*) Multiplication

(*d*) Division

1.20 List two reasons why floating-point number systems are usually normalized.

1.21 In a floating-point system, what quantity determines the maximum relative error in representing a given real number by a machine number?

1.22 (*a*) Explain the difference between the rounding rules "round toward zero" and "round to nearest" in a floating-point system.

(*b*) Which of these two rounding rules is more accurate?

(*c*) What quantitative difference does this make in the unit roundoff ϵ_{mach}?

1.23 In a t-digit binary floating-point system with rounding to nearest, what is the value of the unit roundoff ϵ_{mach}?

1.24 In a floating-point system with gradual underflow (subnormal numbers), is the representation of each number still unique? Why?

1.25 In a floating-point system, is the product of two machine numbers usually exactly representable in the floating-point system? Why?

1.26 In a floating-point system, is the quotient of two nonzero machine numbers always exactly representable in the floating-point system? Why?

1.27 (*a*) Give an example to show that floating-point addition is not necessarily associative.

(*b*) Give an example to show that floating-point multiplication is not necessarily associative.

1.28 Give an example of a number whose decimal representation is finite (i.e., it has only a finite number of nonzero digits) but whose binary representation is not.

1.29 Give examples of floating-point arithmetic operations that would produce each of the exceptional values `Inf` and `NaN`.

1.30 Explain why the cancellation that occurs when two numbers of similar magnitude are subtracted is often bad even though the result may be exactly correct for the actual operands involved.

1.31 Assume a decimal (base 10) floating-point system having machine precision $\epsilon_{\text{mach}} = 10^{-5}$ and an exponent range of ± 20. What is the result of each of the following floating-point arithmetic operations?

(*a*) $1 + 10^{-7}$

(*b*) $1 + 10^{3}$

(*c*) $1 + 10^{7}$

(*d*) $10^{10} + 10^{3}$

(*e*) $10^{10}/10^{-15}$

(*f*) $10^{-10} \times 10^{-15}$

Review Questions

2.1 True or false: If a matrix $\boldsymbol{A}$ is nonsingular, then the number of solutions to the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ depends on the particular choice of right-hand-side vector $\boldsymbol{b}$.

2.2 True or false: If a matrix has a very small determinant, then the matrix is nearly singular.

2.3 True or false: If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.

2.4 True or false: If a matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.

2.5 True or false: An underdetermined system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is an $m \times n$ matrix with $m < n$, always has a solution.

2.6 True or false: The product of two upper triangular matrices is upper triangular.

2.7 True or false: The product of two symmetric matrices is symmetric.

2.8 True or false: The inverse of a nonsingular upper triangular matrix is upper triangular.

2.9 True or false: If the rows of an $n \times n$ matrix $\boldsymbol{A}$ are linearly dependent, then the columns of the matrix are also linearly dependent.

2.10 True or false: A system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if the $m \times n$ matrix $\boldsymbol{A}$ and the augmented $m \times (n+1)$ matrix $[\,\boldsymbol{A} \quad \boldsymbol{b}\,]$ have the same rank.

2.11 True or false: If $\boldsymbol{A}$ is any $n \times n$ matrix and $\boldsymbol{P}$ is any $n \times n$ permutation matrix, then $\boldsymbol{PA} = \boldsymbol{AP}$.

2.12 True or false: Provided row interchanges are allowed, the LU factorization always exists, even for a singular matrix $\boldsymbol{A}$.

2.13 True or false: If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.

2.14 True or false: If a matrix is singular then it cannot have an LU factorization.

2.15 True or false: If a nonsingular symmetric matrix is not positive definite, then it cannot have a Cholesky factorization.

2.16 True or false: A symmetric positive definite matrix is always well-conditioned.

2.17 True or false: Gaussian elimination without pivoting fails only when the matrix is ill-conditioned or singular.

2.18 True or false: Once the LU factorization of a matrix $\boldsymbol{A}$ has been computed to solve a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, then subsequent linear systems with the same matrix but different right-hand-side vectors can be solved without refactoring the matrix.

2.19 True or false: In explicitly inverting a matrix by LU factorization and triangular solution, the majority of the work is due to the factorization.

2.20 True or false: If $\boldsymbol{x}$ is any n-vector, then $\|\boldsymbol{x}\|_1 \geq \|\boldsymbol{x}\|_\infty$.

2.21 True or false: The norm of a singular matrix is zero.

2.22 True or false: If $\|\boldsymbol{A}\| = 0$, then $\boldsymbol{A} = \boldsymbol{O}$.

2.23 True or false: $\|A\|_1 = \|A^T\|_\infty$.

2.24 True or false: If $\boldsymbol{A}$ is any $n \times n$ nonsingular matrix, then $\text{cond}(\boldsymbol{A}) = \text{cond}(\boldsymbol{A}^{-1})$.

2.25 True or false: In solving a nonsingular system of linear equations, Gaussian elimination with partial pivoting usually yields a small residual even if the matrix is ill-conditioned.

2.26 True or false: Since the multipliers in Gaussian elimination with partial pivoting are bounded by 1 in magnitude, the elements of the successive reduced matrices cannot grow in magnitude.

2.27 Can a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ have exactly two distinct solutions?

2.28 Can the number of solutions to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ ever be determined solely from the matrix $\boldsymbol{A}$ without knowing the right-hand-side vector $\boldsymbol{b}$?

2.29 In solving a square system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, which would be a more serious difficulty: that the rows of $\boldsymbol{A}$ are linearly dependent, or that the columns of $\boldsymbol{A}$ are linearly dependent? Explain.

2.30 (*a*) State one defining property of a *singular* matrix $\boldsymbol{A}$.

(*b*) Suppose that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has two distinct solutions $\boldsymbol{x}$ and $\boldsymbol{y}$. Use the property you gave in part *a* to prove that $\boldsymbol{A}$ must be singular.

2.31 Given a nonsingular system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, what effect on the solution vector $\boldsymbol{x}$ results from each of the following actions?

(*a*) Permuting the rows of $[\,\boldsymbol{A} \quad \boldsymbol{b}\,]$

(*b*) Permuting the columns of $\boldsymbol{A}$

(*c*) Multiplying both sides of the equation from the left by a nonsingular matrix $\boldsymbol{M}$

2.32 Suppose that both sides of a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ are multiplied by a nonzero scalar α.

(*a*) Does this change the true solution $\boldsymbol{x}$?

(*b*) Does this change the residual vector $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{Ax}$ for a given $\boldsymbol{x}$?

(*c*) What conclusion can be drawn about assessing the quality of a computed solution?

2.33 Suppose that both sides of a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ are premultiplied by a nonsingular diagonal matrix.

(*a*) Does this change the true solution $\boldsymbol{x}$?

(*b*) Can this affect the conditioning of the system?

(*c*) Can this affect the choice of pivots in Gaussian elimination?

2.34 With a singular matrix and the use of exact arithmetic, at what point will the solution process break down in solving a linear system by Gaussian elimination

(*a*) With partial pivoting?

(*b*) Without pivoting?

2.35 (*a*) What is the difference between partial pivoting and complete pivoting in Gaussian elimination?

(*b*) State one advantage of each type of pivoting relative to the other.

2.36 Consider the following matrix $\boldsymbol{A}$, whose LU factorization we wish to compute using Gaussian elimination:

$$\boldsymbol{A} = \begin{bmatrix} 4 & -8 & 1 \\ 6 & 5 & 7 \\ 0 & -10 & -3 \end{bmatrix}.$$

What will the initial pivot element be if

(*a*) No pivoting is used?

(*b*) Partial pivoting is used?

(*c*) Complete pivoting is used?

2.37 Give two reasons why pivoting is essential for a numerically stable implementation of Gaussian elimination.

2.38 If $\boldsymbol{A}$ is an ill-conditioned matrix, and its LU factorization is computed by Gaussian elimination with partial pivoting, would you expect the ill-conditioning to be reflected in $\boldsymbol{L}$, in $\boldsymbol{U}$, or both? Why?

2.39 (*a*) What is the inverse of the following matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 \\ 0 & m_2 & 0 & 1 \end{bmatrix}$$

(*b*) How might such a matrix arise in computational practice?