Rotation Minimizing Frame (RMF)

Frenet Components

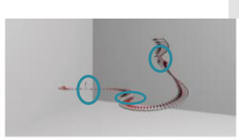
$$\mathbf{t} = \frac{\dot{\mathbf{x}}}{||\dot{\mathbf{x}}||}, \qquad \mathbf{b} = \frac{\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}}{||\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}||}, \qquad \mathbf{m} = \mathbf{b} \times \mathbf{t}$$

$$\kappa = \frac{||\ddot{\mathbf{x}} \wedge \dot{\mathbf{x}}||}{||\dot{\mathbf{x}}||^3}, \quad \tau = \frac{\det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dddot{\mathbf{x}}]}{||\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}||^2}$$

The Problem with Frenet Frames

- Sudden flip about **t** (aka roll) of the normal plane [**m**, **b**]
  - Flips occur at the **inflection points**, where $\kappa = 0$
  - Also described as **indeterminate** behavior of the second difference

"rotation–minimizing" motion $\longrightarrow \omega_{roll} = 0$



(a) The Frenet frame of a spine curve. Only normal vectors are shown.

Definition

- For a curve to have a rotation minimizing frame….
  - There is no instantaneous rotation about the tangent of the spine curve
- Given a vector $\mathbf{r}(u)$ is part of RMF = $(\mathbf{t}, \mathbf{r}, \mathbf{t} \times \mathbf{r})$ if:
  - $\mathbf{r} \cdot \mathbf{t} = \mathbf{0}$ (orthogonal)
  - $\mathbf{r}'(u) = \phi(u)\,\mathbf{t}(u)$ (the change in direction is parallel to the tangent vector)

Why do we care?

- Minimal twist
- Stable at inflection points
- Unique up to an angle constant along the curve (ϕ)

Fig. 1. Sweep surfaces showing moving frames of a deforming curve: the Frenet frames in the first row and the RMF in the second row.

New basis in normal plane

So instead we define our system as:

$$\mathbf{t}(u) := \frac{\mathbf{x}'(u)}{\|\mathbf{x}'(u)\|}, \quad \mathbf{r}(u) := \mathbf{s}(u) \times \mathbf{t}(u), \quad \mathbf{t}(u) \times \mathbf{r}(u) := \mathbf{s}(u)$$

Since $\phi \boldsymbol{t} = (\mathbf{r}' \cdot \mathbf{t})\mathbf{t} = -(\mathbf{r} \cdot \mathbf{t}')\,\mathbf{t} = [\mathbf{t}(u) \times \mathbf{t}'(u)] \times \mathbf{r}(u)$

Due to vector triple product: $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$

$$\left.\begin{matrix} \mathbf{r}'(u) - \phi(u)\,\mathbf{t}(u) = 0 \\ \mathbf{r}(u) \cdot \mathbf{t}(u) = 0 \end{matrix}\right\} \Rightarrow \mathbf{r}'(u) = [\mathbf{t}(u) \times \mathbf{t}'(u)] \times \mathbf{r}(u)$$

$$\mathbf{r}' \cdot \mathbf{t} = \phi\mathbf{t} \cdot \mathbf{t} = \phi$$
$$0 = (\mathbf{r} \cdot \mathbf{t})' = \mathbf{r}' \cdot \mathbf{t} + \mathbf{t}' \cdot \mathbf{r}$$

Or simply, $\mathbf{r}'(s) := \kappa(s)\,\mathbf{b}(s) \times \mathbf{r}(s)$

But what is rotation of a frame along a curve?

$\mathbf{d} := \kappa(s)\boldsymbol{b}(s) + \tau(s)\boldsymbol{t}(s) =: \omega_{Frenet}(s)$, aka Darboux vector

$$\frac{d\mathbf{t}}{ds} := \mathbf{d} \times \mathbf{t}, \quad \frac{d\mathbf{m}}{ds} := \mathbf{d} \times \mathbf{m}, \quad \frac{d\mathbf{b}}{ds} := \mathbf{d} \times \mathbf{b}$$

Frenet Serret Formula

$$\frac{d\mathbf{t}}{ds} := \boldsymbol{\omega} \times \mathbf{t}, \quad \frac{d\mathbf{r}}{ds} := \boldsymbol{\omega} \times \mathbf{r}, \quad \frac{d\mathbf{s}}{ds} := \boldsymbol{\omega} \times \mathbf{s}$$

RMF

To be an RMF – ω satisfies $\omega \bullet \mathbf{t} := 0$

$\omega_{RMF}(s) :=$ ~~$\kappa(s)\mathbf{b}(s)$~~ $= \kappa(s)\mathbf{r}(s)v\mathbf{b}(s)$

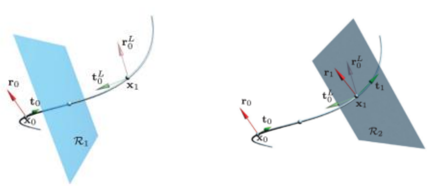


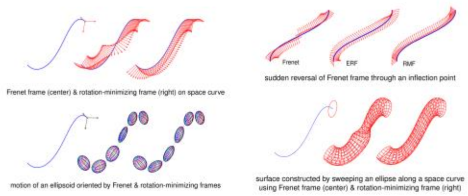
Table I. Algorithm—*Double Reflection*

Input: Points $\mathbf{x}_i$ and associated unit tangent vectors $\mathbf{t}_i$, $i = 0, 1, \ldots, n$.
An initial frame $U_0 = (\mathbf{r}_0, \mathbf{s}_0, \mathbf{t}_0)$.

Output: $U_i = (\mathbf{r}_i, \mathbf{s}_i, \mathbf{t}_i)$, $i = 0, 1, 2, \ldots, n$, as approximate RMF.

Begin
for $i = 0$ to $n - 1$ do
    Begin
1) $\mathbf{v}_1 := \mathbf{x}_{i+1} - \mathbf{x}_i$; /*compute reflection vector of R_1. */
2) $c_1 := \mathbf{v}_1 \cdot \mathbf{v}_1$;
3) $\mathbf{r}_i^L := \mathbf{r}_i - (2/c_1) * (\mathbf{v}_1 \cdot \mathbf{r}_i) * \mathbf{v}_1$; /*compute $\mathbf{r}_i^L = R_1\mathbf{r}_i$. */
4) $\mathbf{t}_i^L := \mathbf{t}_i - (2/c_1) * (\mathbf{v}_1 \cdot \mathbf{t}_i) * \mathbf{v}_1$; /*compute $\mathbf{t}_i^L = R_1\mathbf{t}_i$. */
5) $\mathbf{v}_2 := \mathbf{t}_{i+1} - \mathbf{t}_i^L$; /*compute reflection vector of R_2. */
6) $c_2 := \mathbf{v}_2 \cdot \mathbf{v}_2$;
7) $\mathbf{r}_{i+1} := \mathbf{r}_i^L - (2/c_2) * (\mathbf{v}_2 \cdot \mathbf{r}_i^L) * \mathbf{v}_2$; /*compute $\mathbf{r}_{i+1} = R_2\mathbf{r}_i^L$. */
8) $\mathbf{s}_{i+1} := \mathbf{t}_{i+1} \times \mathbf{r}_{i+1}$; /*compute vector $\mathbf{s}_{i+1}$ of U_{i+1}. */
9) $U_{i+1} := (\mathbf{r}_{i+1}, \mathbf{s}_{i+1}, \mathbf{t}_{i+1})$;
    End
End

Result



- Credit where do:
- Rotation-minimizing frames on space curves — theory, algorithms, applications by Rida T. Farouki, ucsd
  - https://faculty.engineering.ucdavis.edu/farouki/wp-content/uploads/sites/41/2013/02/Rational-rotation-minimizing-frames.pdf
- Wang et al. "Computation of Rotation Minimizing Frames"
  - https://www.researchgate.net/publication/220183621_Computation_of_rotation_minimizing_frames