B-splines and splines

A rough (and not fully precise) characterization of a spline is a *piecewise polynomial* that is as smooth as possible without becoming a single polynomial.

In the following, we define a basis for splines, the collection of B-basis functions, short B-splines. B-splines express a spline in B-form, i.e. as a linear combination of B-splines.

The B-form is particularly well-suited for applications such as graphics and geopmetric modelling, as well as numerical computations. Compared with other representations, B-splines have the advantage built-in continuity between the polynomial pieces – a property that is retained by linear combinations. Further desirable properties become clear as we develop the subject.

(The letter B stands for 'basic' and is denoted by capital letters, for historical reasons even though it is scalar-valued.)

B-spline definition by recursion

Let

$$t_{(i:j)} := t_i, t_{i+1}, \ldots, t_j \tag{1}$$

be a nondecreasing sequence of scalars, i.e. $t_{k+1} \geq t_k$. The scalars are called *knots*. Then the B-spline of degree d is defined recursively as follows.

$$\begin{aligned} B(u|t_{(i:i+1)}) &:= \begin{cases} 1 & \text{if } t_i \leq u < t_{i+1} \\ 0 & \text{otherwise,} \end{cases} \\ B(u|t_{(i:i+d+1)}) &:= \ell(u|_{i,i+d})B(u|t_{(i:i+d)}) \\ &\quad + (1 - \ell(u|_{i+1,i+d+1}))B(u|t_{(i+1:i+d+1)}) \\ \text{where } \ell(u|_{i,j}) &:= \begin{cases} \frac{u-t_i}{t_j-t_i} & \text{if } t_i \neq t_j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

X_5– Verify that for $t_{j+d+1} > t_j$, $B(u|t_{(j:j+d+1)}) > 0$ on the interval (t_j, t_{j+d+1}).

X_5– Show that a B-spline value does not change when the knots and the argument u are all translated by the same amount. Show that a B-spline value does not change when the knots and the argument are all scaled by the same amount. That is, B-spline are invariant under linear reparameterization.

X_{10}– We abbreviate $\mathbf{t} := t_{(j:j+d+1)}$ and denote by $\Pi_{d,\mathbf{t}}$ the set of all piecewise polynomial functions of degree d with breaks at the knots in $\mathbf{t}$. Show that $B(u|\mathbf{t}) \in \Pi_{d,\mathbf{t}}$.

Splines defined

A *spline* s of degree d is a linear combination of B-splines:

$$s(u|t_{(i:i+d+n+1)}) := \sum_{j=i}^{i+n} c_{j:j+d+1} B(u|t_{(j:j+d+1)}), \quad c_{j:k} \in R. \tag{2}$$

We note that there are $n+d+1$ knots for n coefficients. We sometimes abbreviate $c_j := c_{j:j+d+1}$ and $B_j := B(u|t_{(j:j+d+1)})$ when the degree d is understood.

We will see that we can evaluate the spline on the interval $[t_{i+d}, t_{i+n+1}]$. That is, we will need d additional knots on each side of the interval.
A knot sequence

$$\mathbf{t}_Z := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

gives rise to the *uniform* splines (also called *Cardinal Splines*):

$$B(u|i : i + d + 1)) = B(u - i|0 : d + 1)).$$

The recursion for uniform splines simplifies to

$$B(u|0 : d + 1) = uB(u|0 : d) + (d + 1 - u)B(u|1 : d + 1).$$

The knot sequence

$$\mathbf{t}_B := \{\ldots, 0, 0, 0, 1, 1, 1, \ldots\}$$

with $\mu + 1$ zeros and $\nu + 1$ ones yields *polynomials in Bernstein-Bezier form*

$$B_{\mu,\nu}(u) := \binom{\mu+\nu}{\mu}(1-u)^{\mu}u^{\nu} = uB_{\mu,\nu-1}(u) + (1-u)B_{\mu-1,\nu}(u).$$

Spline Evaluation

We can be more efficient than evaluating each B-spline separately and forming linear combinations. The algorithm for obtaining the value directly from the coefficients and the knots is called *de Boor algorithm*. In the special case $t = t_B$ the de Boor algorithm is also called *de Casteljau's* algorithm.

The key to evaluating a spline by recursion is to express coefficients at level $d + 1$ as a linear polynomial in u of two coefficients at level d:

$$\begin{aligned} c_{i:i+d} &:= (1 - \ell(u|_{i,i+d}))\ c_{i-1:i+d} + \ell(u|_{i,i+d})\ c_{i:i+d+1} \\ &=^* \frac{t_{i+d} - u}{t_{i+d} - t_i} c_{i-1:i+d} + \frac{u - t_i}{t_{i+d} - t_i} c_{i:i+d+1}. \end{aligned} \tag{3}$$

Here $=^*$ indicates the case that $t_{i+d} \neq t_i$. While, formally, all $c_{i:j}$ are functions of u, in the following context u will be the parameter of evaluation, and therefore a fixed number so that the expressions in (3) will be constants.

To evaluate the spline $s(u|t_{(i:i+d+n+1)})$ at a value

$$u \in [t_j, t_{j+1}) \subset [t_{i+d}, t_{i+n+1}),$$

we compute $c_{j:j+1}$ by repeatedly applying Equation (3) (cf.). It is convenient

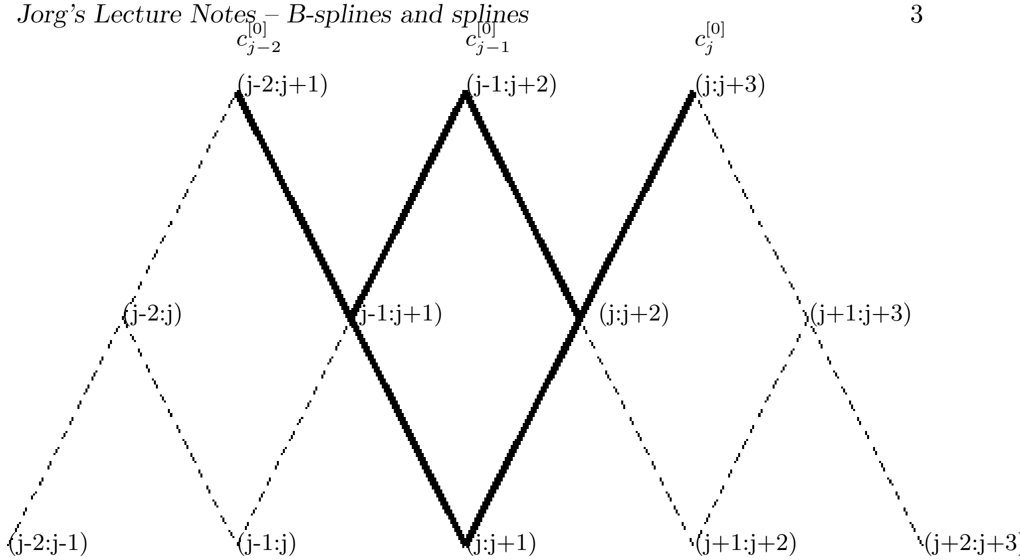

Figure 1: The indices of splines obtained by recursion upwards, i.e. $(a:b)$ for indices a and b corresponds to $B(u|t_{(a:b)})$ and $c_{a:b}$. For the reverse recursion for evaluation, the weights for forming $(a:b)$ are $1-\ell(u|_{a:b}) = \frac{t_b-u}{t_b-t_a}$ and $\ell(u|_{a:b}) = \frac{u-t_a}{t_b-t_a}$ respectively.

to look at a window of relevant knots $t_{j-d:j+d+1}$.

$$\begin{aligned} f(u|t_{(j-d:j+d+1)}) &= \sum_{i=j}^{j+d} c_{i-d:i+1} B(u|t_{(i-d:i+1)}) \\ &= \sum_{i=j}^{j+d-1} c_{i-d+1:i+1} B(u|t_{(i-d+1:i+1)}) = \ldots = \sum_{i=j}^{j} c_{i:i+1} B(u|t_{(i:i+1)}) \\ &= c_{j:j+1} \end{aligned}$$

The last equality follows since splines with two knots are piecewise constant and for $u \in [t_j, t_{j+1}))$, $B(u|t_{(i:i+1)}) = 0$ except that it is 1 when $i = j$.

Example:We evaluate at $u = 0 \in [t_j, t_{j+1}$, a spline defined by $d = 2$ and

$$\mathbf{t} = \begin{bmatrix} -3 & -2 & -1 & 1 & 5 & 6 \end{bmatrix}, \quad c = \begin{bmatrix} 48 & 12 & 24 \end{bmatrix}.$$

Therefore $t_j = -1$ and $c_{j-2:j+1} = 48$. We compute

$$\begin{aligned} \ell(u|_{j-1:j+1}) &= \frac{u - t_{j-1}}{t_{j+1} - t_{j-1}} = \frac{0-(-2)}{1-(-2)} = \frac{2}{3} \\ \ell(u|_{j:j+2}) &= \frac{u - t_j}{t_{j+2} - t_j} = \frac{0-(-1)}{5-(-1)} = \frac{1}{6} \\ \ell(u|_{j:j+1}) &= \frac{u - t_j}{t_{j+1} - t_j} = \frac{0-(-1)}{1-(-1)} = \frac{1}{2} \end{aligned}$$

The schema then reads

$$\begin{array}{rccccc}
c_{j-2:j+1}: & 48 & & 12 & & 24 \\
\ell(u|_{j:j+2}): & & 2/3 & & 1/6 & \\
c_{j-1:j+1}: & & 24 & & 14 & \\
\ell(u|_{j:j+1}): & & & 1/2 & & \\
c_{j:j+1}: & & & 19 & &
\end{array}$$

In the example, we need 6 knots and 3 coefficients to have one interval, $[-1..1]$. The define the three degree 2 B-splines that have support (are non-zero) on the interval.

X_5– Using the data of the example, associate each coefficient c_j with its *Greville abscissa* defined by

$$t_j^* := \frac{1}{d}\sum_{i=1}^{d} t_{j+i}$$

and draw the control polygon and the de Boor evaluation.
X_3– Show that for $t_j \le u \le t_{j+1}$ all $0 \le \ell(u|_{i:j}) \le 1$.
X_3– Check that the algorithm is well-defined for multiple knots.

Differentiation

The spline $s(u|t_{(i:i+d+n+1)}) := \sum_{j=i}^{i+n} c_{j:j+d+1} B(u|t_{(j:j+d+1)})$ of degree d has the derivative

$$s'(u|t_{(i:i+d+n+1)}) := \sum_{j=i+1}^{i+n} c'_{j:j+d} B(u|t_{(j:j+d)}), \tag{4}$$

where

$$c'_{j:j+d} := d\,\frac{c_{j:j+d+1} - c_{j-1:j+d}}{t_{j+d} - t_j}$$

Multiplication

X_{20}– What challenges do you see in deriving the coefficients and knots of the spline that is the product of two given splines? (Conversion to BB-form and back helps, but here the structure is asked)

Integration

The spline $s(u|t_{(i:i+d+n-1)}) := \sum_{j=i}^{i+n} c_{j:j+d} B(u|t_{(j:j+d)})$, has the antiderivative

$$\sum_{j=i}^{i+n} c^*_{j:j+d+1} B(u|t_{(j:j+d+1)}), \qquad c^*_{j:j+d+1} := \text{const} + \begin{cases} \sum_{i=j_0}^{j} c_{i:i+d}\frac{t_{i+d}-t_i}{d} \\ \sum_{i=j}^{j_0-1} c_{i:i+d}\frac{t_{i+d}-t_i}{d} \end{cases} \tag{5}$$

where j_0 and the constant are arbitrary (j_0 separates two subsets of indices).

Continuity

The knots t_j and t_{j+1} may agree. Such coalescing of knots decreases the otherwise guaranteed continuity of the B-spline and hence of the spline $s_{d,\mathbf{t}}$ as follows. If m is the multiplicity of the knot t_j, i.e. $\ldots < t_j = \ldots = t_{j+m-1} < \ldots$ then $s_{d,\mathbf{t}}$ is at least

$$k = d \text{ - } m$$

times continuously differentiable at t_j.

Reproduction

Marsden's identity

$$(u-\tau)^d = \sum_{j=0}^{d} a_{j:j+d}(\tau) B(u|\mathbf{t}), \quad \mathbf{t} := t_{(j:j+d+1)},$$
$$a_{j:j+d}(\tau) := (t_{j+1}-\tau)\cdots(t_{j+d}-\tau)$$

shows what coefficients $c_{\mathbf{t}}$ the spline $\sum_j c_{\mathbf{t}} B(u|\mathbf{t})$ has to have so that so that the spline reproduces a given polynomial $p := \sum_{j=0}^{k}(u-\tau)^j$.

Differentiating i times with respect to τ and dividing by $d!$ yields

$$\frac{(u-\tau)^{d-j}}{(d-j)!} = \frac{(-1)^j}{d!} \sum_i B(u|t_{(i:I+d+1)}) D^j a_{i:i+d}(\tau).$$

Taylor expansion gives the coefficients,

$$c_i = \lambda(d,i)p := \sum_{j=0}^{d} \frac{(-1)^j (D^j a_{i:i+d})(\tau)}{d!} (D^{d-j}p)(\tau).$$

The $\lambda(d,i)$ are called *dual functionals.*
X_5– Show that for fixed degree d and any knot sequence $(t_i)_{i=-\infty..\infty}$

$$\sum_j B(u|t_{(j:j+d+1)}) = 1. \tag{6}$$

X_5– Show for any linear function p and Greville abscissae t_i^*

$$\sum_j B(u|t_{(j:j+d+1)}) p(t_j^*) = p. \tag{7}$$

X_{10}– Show that for $\tau \in [x_j, x_{j+d+1})$

$$\lambda(d,i)(\sum_j B(u|t_{(j:j+d+1)}) c_j) = c_i. \tag{8}$$

Of course splines are not usually used to reproduce polynomials but to approximate (or even interpolate) a sequence of points.

Interpolation

(c.f. cubic spline interpolation)

The *Schoenberg-Whittney interpolation theorem*: given a vector of interpolation points u_j and values $f(u_j)$, there always exists a spline of degree d that is $d-1$ times continuously differentiable and interpolates the data.
A necessary and sufficient condition for the banded system of interpolation constraints to be solvable is $t_j < u_j < t_{j+d+1}$.

Stability and local well-conditioning of a spline basis

Small spline function implies small spline coefficients ($c_j \neq 0$ for some j):

$$2^{-d} \max_i \|c_i\| < \| \sum c_j B_j \|.$$

$\| \sum c_j B_j \|_\infty \leq \max_i \|c_i\|$ follows from $\sum B_j = 1$.

Knot insertion and subdivision.

Let $C_{c,t}$ be the control polygon of the spline s. Then

$$\begin{aligned} &\sup_u |s(u|t) - C_{c,t}(u)| \\ &\qquad \leq \mathit{const} \sup_u |D^2 s(u)|| \sup_j (t_{j+1} - t_j)|^2. \end{aligned}$$

That is, the control polygon *converges with quadratic error* to s as the distance between the knots decreases. Inserting a knot requires generating new coefficients without changing the spline. The correct way to do this when the knot $\hat{t}$ is inserted into the sequence t is to choose the *new coefficient sequence* as follows.

$$\hat{c}_j := \begin{cases} c_j, & t_{j+d} \leq \hat{t}; \\ (1 - \ell_{d,j}(\hat{t}))c_{j-1} + \ell_{d,j}(\hat{t})c_j, & t_j < \hat{t} < t_{j+d}; \\ c_{j-1}, & \hat{t} \leq t_j \end{cases}$$

Example: Given the spline defined by

$$\begin{aligned} t &= \begin{bmatrix} -3 & -2 & -1 & 1 & 5 & 6 \end{bmatrix}, \\ c &= \begin{bmatrix} 48 & 12 & 24 \end{bmatrix}. \end{aligned}$$

insert a new knot $\hat{t} = 0$. We compute

$$\begin{array}{cccccccc} c & 48 & & 12 & & 24 \\ w_2: & & 2/3 & & 1/6 & \\ \hat{c}: & 48 & 24 & 14 & & 24 \end{array}$$

The *Greville abscissae* are

$$\begin{aligned} t^* &= -1.5, 0, 3, \quad \text{before insertion and} \\ t^* &= -1.5, -.5, .5, 3, \quad \text{after insertion.} \end{aligned}$$

X_{10}– Given the spline *curve* defined by

$$t = [0, 0, 0, 0, 1, 3, 5, 5, 5, 5],$$
$$c = \begin{bmatrix} -1 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

compute the new control polygon after insertion of the new knot $\hat{t} := 2$. (Hints: there are knots $x_0, ..., x_9$ and 6 intervals, hence $9 = 6 + d$. For a curve we do not display the Greville abscissae but only the (x, y) pairs.)
X_{10}– Given the spline function of degree 3 with coefficients c and $t = t_{2Z} = \ldots, -2, 0, 2, \ldots$ give the coefficient sequence after insertion of knots at every odd integer; that is after changing the knot sequence to t_Z.

The convex hull property

X_3– Show that for $t_j \leq u < t_{j+1}$, $s(u)$ is a convex combination of $c_{j-d}, \ldots, c_j$.

Variation diminuition

The number of strong sign changes in any increasing sequence of values $p_d(x_1), \ldots, p_d(x_r)$ of the spline is less than the number of strong sign changes in the sequence of coefficients c_i. This is often written as

$$S^-(s) \leq S^-(c),$$

or $S^-(s(t_1), \ldots, s(t_n)) \leq S^-(c_1, \ldots, c_n)$ for t strictly increasing. Proof: insert t_i into the knot sequence to obtain new coefificnets by convex averaging.

Shape preservation

A spline crosses any straight line no more often than its control polygon. Proof: $S(s - \ell) \leq S(c - \ell)$.
X_3– Show that shape preservation implies that a spline is *monotone* if its control polygon is monotone. X_3– Show that shape preservation implies that a spline is *convex* if its control polygon is convex.

Alternative definitions of B-splines

(that allow the definition of splines in several dimensions)

1. Hermite-Genocchi formula: B-spline as distribution.

$$\int_R B_d(x, u)\phi(u)du$$
$$= d! \int_{\Delta(d)} \phi(\lambda x)d\lambda$$

where $\Delta(d)$ is the d-*simplex* spanned by
$e_0, e_1, \ldots, e_d$, $e_0 = 0$, e_i the ith unit vector.

2. Geometric interpretation. With T a d-simplex such that the first component of each vertex is one of the knots x_j and π the projection $\mathbb{R}^d \mapsto \mathbb{R}$: $\pi(x_1, x_2, \ldots, x_d) = x_1$. Then

$$B_d(x, u) := \mathrm{vol}_{d-1}(T \cap (\pi^{-1}(u)))/\mathrm{vol}_d(T).$$