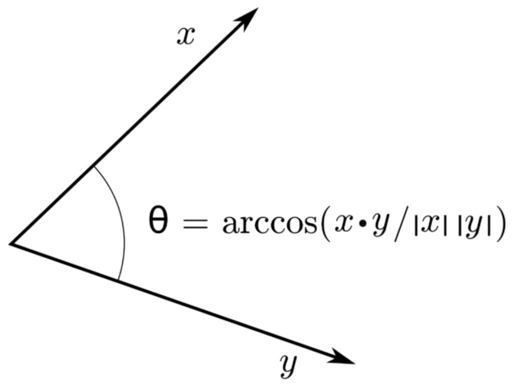
Dot Product

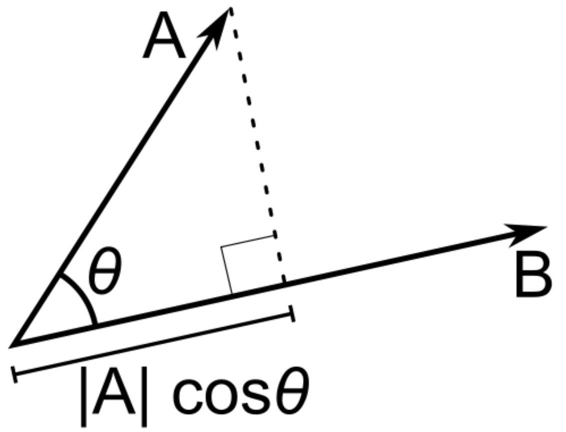
- $\vec{a}, \vec{b} \in \mathbb{R}^n, \vec{a} \cdot \vec{b} = \sum_{i=1}^{n}(a_i \cdot b_i) \in \mathbb{R}$.
- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.
  - θ **convex** angle between the vectors.
- Squared norm of vector: $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$.
- Alternative notation: $\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle$
- Matrix multiplication $C = A \cdot B \Leftrightarrow c_{ij} = \langle A_{i,\cdot}, B_{\cdot,j} \rangle$
- Note: $\langle \vec{a}, \vec{a} \rangle$ is always non-negative.
  - $\langle \vec{a}, \vec{b} \rangle$ - measure similarity (angle)
  - $\langle \vec{a}, \vec{a} \rangle$ - measures length.



Dot Product

- A geometric interpretation: the part of $\vec{a}$ which is parallel to a unit vector in the direction of $\vec{b}$.
  - And vice versa!
  - Projected vector: $\overrightarrow{a_{\parallel}} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|} \vec{b}$.

- The part of $\vec{b}$ orthogonal to $\vec{a}$ has no effect!



Linear Transformations

- Represented as a matrix: $y = Mx, y \in \mathbb{R}^m, x \in \mathbb{R}^n, M \in \mathbb{R}^{m \times n}$
- In fact, a stack of dot products:
- $y = \begin{pmatrix} \langle M_{1,.}, x \rangle \\ \vdots \\ \langle M_{n,.}, x \rangle \end{pmatrix}$
- Geometric interpretation: transforming x from an axis system on the columns of M to the canonical axis system.
- Canonical axis system: $(1,0,0,0, \dots)$ etc.
- When $m = n$, and the matrix is full-rank, it is a change of coordinates.

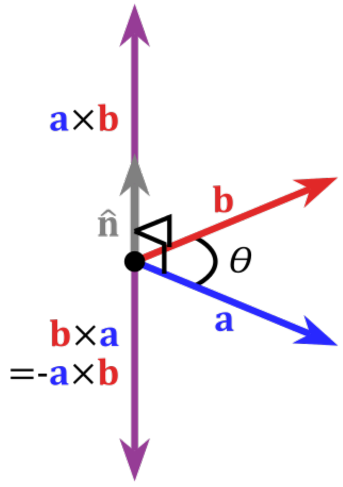
Special Linear Transformations

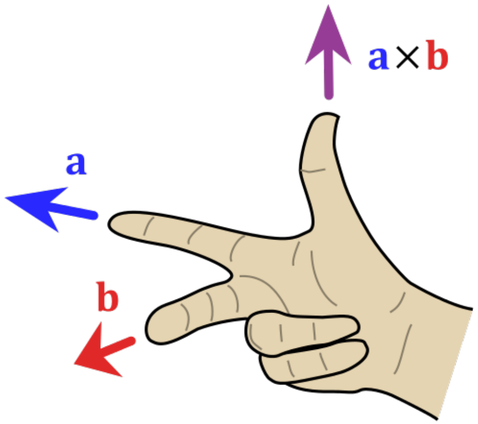
- Rotation matrix in $\mathbb{R}^2$: $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$.
- What you learned: rotates a point $p = (x, y)$ by angle θ in CCW direction.
- Alternative interpretation: transforms p from its representation in a rotated axis system to its representation in the canonical one.
    - Watch chalkboard!

Cross Product

- Typically defined only for $\mathbb{R}^3$.
- $\vec{a} \times \vec{b} = \left(a_y b_z - a_z b_y, b_x a_z - b_z a_x, a_x b_y - a_y\right) \in \mathbb{R}^3$.
- Or more generally:

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$$



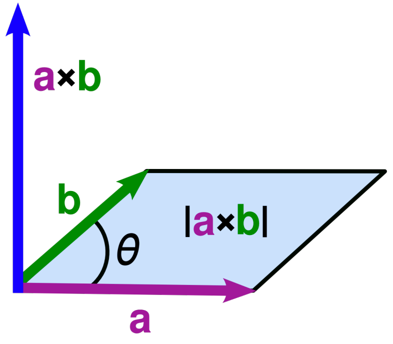


Cross Product

- The result vector is orthogonal to both vectors.
  - Direction: Right-hand rule.
  - Normal to the plane spanned by both vectors.
- Its magnitude is $\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta$.
  - Parallel vectors ⇔ cross product zero.
- The part of $\vec{b}$ parallel to $\vec{a}$ has no effect on the cross product!
- Geometric interpretation: axis of shortest rotation between $\vec{a}$ and $\vec{b}$.

$$\vec{b} = \mathrm{R}_{\vec{a}\times\vec{b}}(\theta)\vec{a}$$

- Another geometric interpretation: $\left|\vec{a}\times\vec{b}\right| = A_{parallelogram}$.



Symmetric Bilinear Maps

- Also denoted as "2-tensors".
- $M: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, M(\vec{u}, \vec{v}) = c.$
- Take two vectors into a scalar.
- Symmetry: $M(\vec{u}, \vec{v}) = M(\vec{v}, \vec{u})$
- Linearity: $M(a\vec{u} + b\vec{w}, \vec{v}) = aM(\vec{u}, \vec{v}) + bM(\vec{w}, \vec{v}).$
  - The same for $\vec{v}$ for symmetry.
- Can be represented by symmetric $n \times n$ matrices: $\mathrm{c} = \vec{u}^T M \vec{v}.$

Metric

- M is positive definite if for every $\vec{u}$, $M(\vec{u}, \vec{u}) > 0$.
  - Consequently, negative-definite, positive semidefinite (≥ 0).
- Interpretation:
  - M is a generalized dot product, or a metric.
  - The original dot product: simply $M = I_{nxn}$.
  - It's only a true metric (=non-negative) if indeed M is PSD.
- Often notated $\langle \vec{a}, \vec{b} \rangle_M$.
  - Then, $\langle \vec{a}, \vec{a} \rangle_M$ is "the squared length of $\vec{a}$ in the metric of M".

Functions of Several Variables

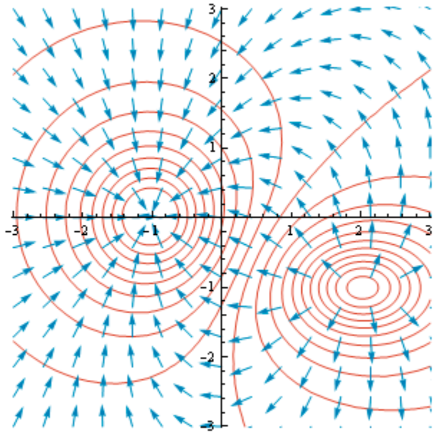
- A single function of several variables:

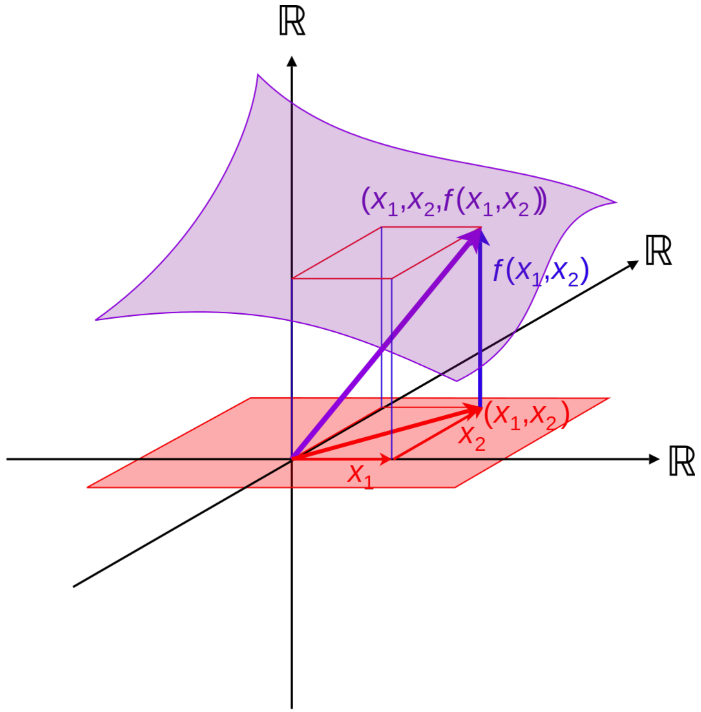
$$f\colon \mathbb{R}^n \to \mathbb{R}, f(x_1, x_2, \cdots, x_n) = y.$$

- Partial derivative vector, or gradient, is a vector:

$$\nabla f = \left(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n}\right)$$

- In the direction of steepest ascent.





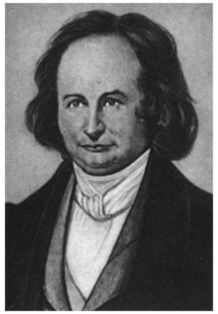
Multi-Valued Functions

- A vector-valued function of several variables:

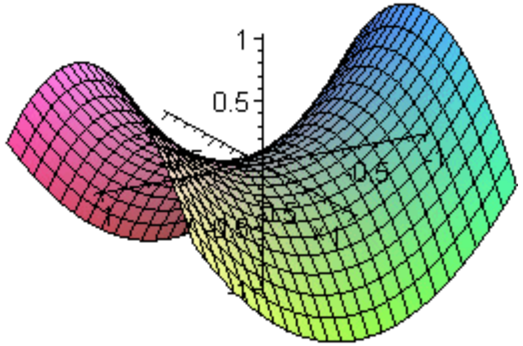
$$f: \mathbb{R}^n \to \mathbb{R}^m, f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m).$$

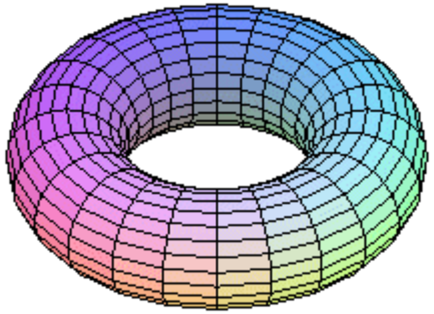
- Can be viewed as a change of coordinates, or a mapping.
  - Recall: Linear functions $\Leftrightarrow \mathbb{R}^{m \times n}$ matrices.

- The derivatives form a matrix, denoted as the Jacobian:



$$\nabla f = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \vdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & & \cdots \\ \frac{\partial y_n}{\partial x_1} & \vdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$





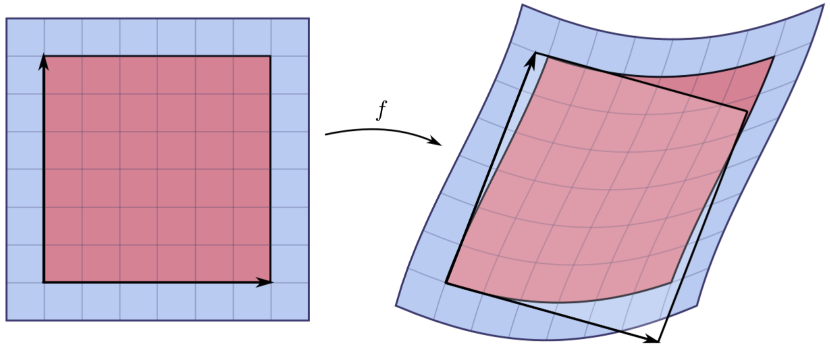
https://www.math.duke.edu/education/ccp/materials/mvcalc/parasurfs/para1.html

$f: \mathbb{R}^n \to \mathbb{R}^m, f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m)$

Jacobian Measures Deformation

- Consider two points $p, p + \Delta p$, where Δp is very small.
- Taylor series:

$$f(p + \Delta p) \approx f(p) + J_f \Delta p$$



- 1^{st}-order linear approximation.

- Original infinitesimal squared length: $\langle \Delta p, \Delta p \rangle$.
- Target length after map:

$$\langle f(p + \Delta p) - f(p), f(p + \Delta p) - f(p) \rangle \approx$$
$$\langle J_f \Delta p, J_f \Delta p \rangle = \langle \Delta p, \Delta p \rangle_M$$

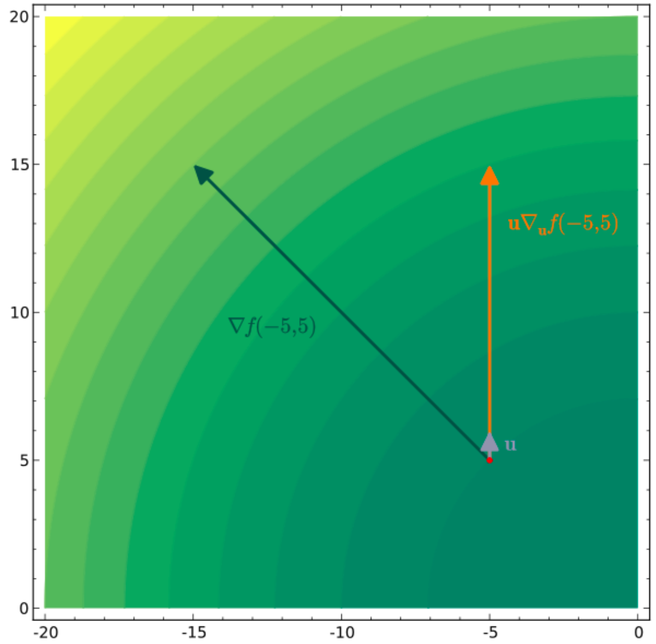
Where $M = \left({J_f}^T \cdot J_f \right)$, a symmetric bilinear form!

- Interpretation: J_f encodes the change of lengths, or deformation.

Directional Derivative

- The change in function u in the (unit) direction $\hat{d}$:
$$\nabla_d u = \langle \nabla u \, , d \rangle$$

- Formally: a map $\nabla_d : \mathbb{R}^n \to \mathbb{R}$ between direction $\hat{d}$ and scalar $\langle \nabla u \, , d \rangle$



Vector Fields in 3D

- A vector-valued function assigning a vector to each point in space: $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3, g(\vec{p}) = \vec{v}$.
- Physics: velocity fields, force fields, advection, etc.
- Special vector fields:
  - Constant
  - Rotational
  - Gradient fields of scalar functions: $\vec{v} = \nabla f$.

http://vis.cs.brown.edu/results/images/Laidlaw-2001-QCE.011.html