A Design of Topological Predicates for Complex Crisp and Fuzzy Regions

Markus Schneider

FernUniversität Hagen, Praktische Informatik IV
D-58084 Hagen, Germany
markus.schneider@fernuni-hagen.de

Abstract. For a long time topological predicates between spatial objects have been a main area of research on spatial data handling, reasoning, and query languages. But these predicates still suffer from two main restrictions: first, they are only applicable to simplified abstractions of spatial objects like single points, continuous lines, and simple regions, as they occur in systems like current geographical information systems and spatial database systems. Since these abstractions are usually not sufficient to cope with the complexity of geographic reality, their generalization is needed which especially has influence on the nature and definition of their topological relationships. This paper gives a formal definition of complex crisp regions, which may consist of several components and which may have holes, and it especially shows how topological predicates can be defined on them. Second, topological predicates so far only operate on crisp but not on fuzzy spatial objects which occur frequently in geographical reality. Based on complex crisp regions, this paper gives a definition of their fuzzy counterparts and shows how topological predicates can be defined on them.

1 Introduction

Representing, storing, quering, and manipulating spatial information is important for many non-standard database applications. Specialized systems like geographical information systems (GIS), spatial database systems, and image database systems to some extent provide the needed technology to support these applications. For these systems the development of formal models for spatial objects and for topological relationships between these objects is a topic of great importance and interest, since these models exert a great influence on the efficiency of spatial systems and on the expressiveness of spatial query languages.

In recent years, significant achievements have been made on the design of topological predicates for spatial objects with precisely defined boundaries, so-called *crisp* spatial objects. However, the structure of spatial objects upon which current topological predicates operate is restricted and not sufficient to cope with the complexity of geographic reality. For spatial regions this means that at most simple regions and topological predicates between them can be found in current GISs and spatial database systems. Only very few approaches exist for more complexly structured regions. General topological predicates on complex regions possibly consisting of several components and possibly having holes have so far not been designed. But in real applications complex regions are by far more common than simple ones. It is one of the goals of this paper to give a definition of complex crisp regions and to provide topological predicates for them.

Additionally, the current mapping of spatial phenomena of the real world to exclusively crisp spatial objects turns out to be an insufficient abstraction for many spatial applications, because the feature of *spatial vagueness* or *spatial fuzziness* is inherent to many geographic data [2]. Spatial fuzziness captures the property of many spatial objects in reality which do not have sharp boundaries or whose boundaries cannot be precisely determined. Examples are natural, social, or cultural phenomena like land features with continuously changing properties (such as population density, soil quality, vegetation, pollution, temperature, air pressure), oceans, deserts, English speaking areas, or mountains and valleys. We will designate this kind of entities as *fuzzy* spatial objects.

The definition of topological predicates on fuzzy spatial objects in general and fuzzy regions in particular is currently an open problem. For two fuzzy regions A and B we would like to be able to pose and answer queries like

- Do regions A and B overlap *a little bit*?
- Determine all pairs of regions that *nearly completely* overlap.
- Does region A *somewhat* contain region B?
- Which regions lie *quite* inside B?

Section 2 discusses related work. In Section 3, we present a formal model of complex crisp and fuzzy regions. For fuzzy regions we use a representation that reduces these objects to collections of so-called crisp α-level regions. This enables us to transfer our whole formal framework (and later all the well known implementation methods available) for crisp regions to fuzzy regions. In Section 4, based on well known topological relationships for *simple* crisp regions, in a bottom-up strategy we first define topological predicates for simple crisp regions with holes and afterwards for complex crisp regions with additional multiple components. Section 5 presents an approach for designing topological predicates for fuzzy regions. Finally, Section 6 draws some conclusions.

2 Related Work

This section summarizes some related work on the definition and representation of crisp and fuzzy regions (Section 2.1) and on the design and definition of binary topological predicates between regions (Section 2.2).

2.1 Crisp and Fuzzy Regions

In the past, a number of data models and query languages for *crisp* spatial data have been proposed with the aim of formulating and processing spatial queries in databases (see [12] for a survey). *Spatial data types* like *point*, *line*, or *region*, that are the central concept of these approaches, provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. However, data models expressing spatial vagueness are rare. *Exact models* [4, 8, 11] transfer type systems for spatial objects with sharp boundaries to objects with unclear boundaries. The approaches in [4, 11] extend the indeterminate boundary of a region into a boundary zone, called *broad boundary*, which is situated around the region. The concept of *vague regions* [8] generalizes these approaches in the sense that such a region can be a pair of arbitrarily located, disjoint crisp regions. The *kernel region* describes the area which definitely belongs to the vague region. The *boundary region* describes the area for which it is not sure whether it or parts of it belong to the vague region or not. *Models based on rough sets* [16] work with lower and upper approximations of spatial objects. *Models based on fuzzy sets* [1, 13, 14] model the vagueness resulting from the imprecision of the meaning of a concept. A concept like 'ocean' or 'Southern England' cannot be modeled with crisp but with fuzzy means. *Fuzzy spatial data types* defined on an abstract (Euclidean space) and on a discrete (*grid partition*) geometric basis are introduced in [13, 14].

2.2 Crisp and Fuzzy Topological Predicates

Our definitions are based on the so-called 9-*intersection model* [6] from which a complete collection of mutually exclusive topological relationships can be derived for each combination of spatial types. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object A with the corresponding components of another object. Each intersection is tested for the topologically invariant criteria of emptiness and non-emptiness. $2^9 = 512$ different configurations are possible from which only a certain subset makes sense depending on the combination of spatial objects just considered.

A restriction of the 9-intersection model with respect to regions is that regions must be homeomorphic to the closed disc, that is, they must be connected and are not allowed to have holes. These regions are usually called *simple regions*. For two simple regions, eight meaningful configurations have been identified which lead to the eight predicates of the set T_{sr} = {*disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, and *coveredBy*}. Each predicate is uniquely determined so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness.

In this paper we aim at a formal definition of topological predicates for crisp and fuzzy complex regions with multiple parts and possibly with holes. It is surprising that topological predicates on crisp complex regions have so far not been defined. In [3] the so-called TRCR (Topological Relationships for Composite Regions) model only allows sets of disjoint simple regions without holes. In [7] only topological relationships of simple regions with holes are considered. Topological predicates on fuzzy spatial objects, let them be simple or complex, have so far not been defined.

3 A Model for Crisp and Fuzzy Complex Regions

In this paper we only consider topological predicates that operate on regions. Hence, in this section, we first clarify the structure and semantics of region objects. We begin with an abstract model for very general crisp complex regions, which results in a spatial data type *region*. Based on this specification, we define a data type *fregion* representing a fuzzy region as a collection of crisp regions with special properties. This representation is later used as operand of fuzzy topological predicates.

3.1 Modeling Crisp Regions

Our definition of regions is based on point set theory and point set topology [9]. Regions are embedded into the two-dimensional Euclidean space $\mathbb{R}^2$ and are thus point sets. Unfortunately, the use of pure point set theory for their definition causes problems. If regions are modeled as arbitrary point sets, they can suffer from undesired geometric anomalies. These degeneracies relate to isolated or dangling line and point features as well as missing lines and points in the form of cuts and punctures. A process called *regularization* [15] avoids these anomalies.

We briefly summarize some needed concepts from point set topology. Let X be a set and $T \subseteq 2^X$. The pair (X, T) is called a *topological space* if the following three axioms hold: (i) $U, V \in T \Rightarrow U \cap V \in T$, (ii) $S \subseteq T \Rightarrow \bigcup_{U \in S} U \in T$, and (iii) $X \in T$, $\varnothing \in T$. T is called a *topology* for X. The elements of T are called *open sets* and their complements in X are called *closed sets*. Several operations identify certain parts of a set. Let $S \subseteq X$. The *interior* of S is defined as the union of all open sets that are contained in S and is denoted by S°. The *closure* of S is defined as the intersection of all closed sets that contain S and is denoted by $\overline{S}$. The *exterior* of S is the union of all open sets that are not contained in S, that is, $S^- := (X - S)^\circ$. The *boundary* of S is the intersection of the closure of S and the closure of the complement of S, that is, $\partial S := \overline{S} \cap \overline{X - S}$. Furthermore, we have $\overline{S} = S^\circ \cup \partial S$.

In our case $X := \mathbb{R}^2$ holds. The concept of regularity defines a point set S as *regular closed* if $S = \overline{S^\circ}$. We define a *regularization function* reg_c which associates a set S with its corresponding regular closed set as $reg_c(S) := \overline{S^\circ}$. The effect of the *interior* operation is to eliminate dangling points, dangling lines, and boundary parts. The effect of the *closure* operation is to eliminate cuts and punctures by appropriately supplementing points and to add the boundary. We are now already able to give a general definition of a type for complex crisp regions:

$$region = \{R \subseteq \mathbb{R}^2 \mid R \text{ is bounded and regular closed}\}$$

In fact, this very "structureless" definition models complex crisp regions possibly consisting of several components and possibly having holes. But since the topological predicates of the 9-intersection model only work on simpler regions, we have to take a more fine-grained and structured view of regions.

A *simple region* is a bounded, regular closed set homeomorphic (that is, topologically equivalent) to a two-dimensional closed disc[1]. This, in particular, means that a simple region has a connected interior, a connected boundary, and a single connected exterior. Hence, it does not consist of several components, and it does not have holes.

The concept of a hole is topologically not inferable since point set topology does not distinguish between outer exterior and inner exteriors of a set. This requires an explicit and constructive definition of a region containing holes and a use of the topological predicates for simple regions. Let $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}, k, n \in \mathbb{N}, k \leq n$, be a total, injective mapping, and let $\{F_0, \ldots, F_n\}$ be a set of simple regions. The regular set $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ is called a *simple region with holes* or a *face*, and $F_1, \ldots, F_n$ are called *holes* (Figure 1c) iff

(i) $\forall\, 1 \leq i \leq n : contains(F_0, F_i) \;\vee\; (covers(F_0, F_i) \;\wedge\; |F_0 \cap F_i| = 1)$
(ii) $\forall\, 1 \leq i < j \leq n : disjoint(F_i, F_j) \;\vee\; (meet(F_i, F_j) \;\wedge\; |F_i \cap F_j| = 1)$
(iii) $\nexists\, \{\pi(1), \ldots, \pi(k)\} \subseteq \{1, \ldots, n\} : meet(F_0, F_{\pi(1)}) \wedge$
$meet(F_{\pi(1)}, F_{\pi(2)}) \;\wedge\; \cdots \;\wedge\; meet(F_{\pi(k-1)}, F_{\pi(k)}) \;\wedge\; meet(F_{\pi(k)}, F_0)$
(iv) $\nexists\, \{\pi(1), \ldots, \pi(k)\} \subseteq \{1, \ldots, n\} : meet(F_{\pi(1)}, F_{\pi(2)}) \wedge$
$meet(F_{\pi(2)}, F_{\pi(3)}) \;\wedge\; \cdots \;\wedge\; meet(F_{\pi(k-1)}, F_{\pi(k)}) \;\wedge\; meet(F_{\pi(k)}, F_{\pi(1)})$

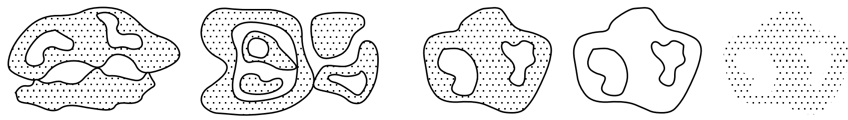
The first two conditions allow a hole within a face to touch the boundary of F_0 or of another hole in at most a single point. This is necessary in order to achieve closure under the geometric operations *union*, *intersection*, and *difference* (see also [10, 12]). For example, subtracting a face A from a face B may lead to such a hole in B. On the other hand, to allow two holes to have a partially common border makes no sense because then adjacent holes could be merged to a single hole by eliminating the common border (similarly for adjacency of a hole with the boundary of F_0). The third condition prevents the formation of "open hole chains" where any two subsequent holes meet and both the first and the last hole touch F_0. The fourth condition prevents the formation of "closed hole chains" within the face where any two subsequent holes meet and both the first and the last hole meet. All four conditions together ensure uniqueness of representation, that is, there are no two different interpretations of the point set describing a face. Hence, a face is atomic and cannot be decomposed into two or more faces. For example, the configuration shown in Figure 1a must be interpreted as two faces with two holes and not as a single face with four holes.

Let $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ be a simple region with holes $F_1, \ldots, F_n$. Then the boundary and the interior of F are given as follows (Figures 1d and 1e):

(i) $\partial F = \bigcup_{i=0}^{n} \partial F_i$
(ii) $F^\circ = F_0^\circ - \bigcup_{i=1}^{n} F_i$

Let $\{F_1, \ldots, F_n\}$ be a set of simple regions with holes, that is, faces. The regular set $F = \bigcup_{i=1}^{n} F_i$ is called a (*complex*) *region* iff

[1] $D(x, \epsilon)$ denotes a two-dimensional closed disc with center x and radius ϵ iff $D(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$ where d is a metric on X.



(a) (b) (c) (d) (e)

Fig. 1. Unique representation of a face (a), a complex region with five faces (b), a simple region with two holes (c), its boundary (d), and its interior (e).

(i) $\forall\, 1 \leq i < j \leq n : F_i^\circ \cap F_j^\circ = \varnothing$

(ii) $\forall\, 1 \leq i < j \leq n : \partial F_i \cap \partial F_j = \varnothing \;\vee\; |\partial F_i \cap \partial F_j|$ is finite)

Figure 1b shows an example of a region with five faces. The definition requires of a face to be disjoint to another face, or to meet another face in one or several single boundary points, or to lie within a hole of another face and possibly share one or several single boundary points with the boundary of the hole. Faces having common connected boundary parts with other faces or holes are disallowed. The argumentation is similar to that for the face definition.

Let $F = \bigcup_{i=1}^n F_i$ be a region with faces $\{F_1, \ldots, F_n\}$. Then the boundary of F is given as $\partial F = \bigcup_{i=1}^n \partial F_i$, and the interior of F is given as $F^\circ = \bigcup_{i=1}^n F_i^\circ = F - \partial F$.

3.2 Some Basic Concepts of Fuzzy Set Theory

Fuzzy set theory [17] is an extension and generalization of Boolean set theory. Let X be a classical (crisp) set of objects. Membership in a classical subset A of X can then be described by the *characteristic function* $\chi_A : X \to \{0,1\}$ such that for all $x \in X$ holds $\chi_A(x) = 1$ if and only if $x \in A$ and $\chi_A(x) = 0$ otherwise. This function can be generalized such that all elements of X are mapped to the real interval [0,1] indicating the *degree of membership* of these elements in the set in question. We call $\mu_{\tilde{A}} : X \to [0,1]$ the *membership function* of $\tilde{A}$, and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ is called a *fuzzy set* in X. Elements $x \in X$ that do (not) belong to $\tilde{A}$ get the membership value $\mu_{\tilde{A}}(x) = 1$ (0).

A [*strict*] *α-cut* or [*strict*] *α-level set* of a fuzzy set $\tilde{A}$ for a specified value α is the crisp set $A_\alpha\,[A_\alpha^*] = \{x \in X \mid \mu_{\tilde{A}}(x) \geq [>]\, \alpha \;\wedge\; 0 \leq \alpha \leq [<]\, 1\}$. The strict α-cut for $\alpha = 0$ is called *support* of $\tilde{A}$, i.e., $supp(\tilde{A}) = A_0^*$. For a fuzzy set $\tilde{A}$ and $\alpha, \beta \in [0,1]$ we obtain $X = A_0$ and $\alpha < \beta \;\Rightarrow\; A_\alpha \supseteq A_\beta$. The set of all levels $\alpha \in [0,1]$ that represent distinct α-cuts of a given fuzzy set $\tilde{A}$ is called the *level set* $\Lambda_{\tilde{A}}$ of $\tilde{A}$: $\Lambda_{\tilde{A}} = \{\alpha \in [0,1] \mid \exists\, x \in X : \mu_{\tilde{A}}(x) = \alpha\}$.

3.3 Modeling Fuzzy Regions

A "structureless" definition of fuzzy regions in the sense that only "flat" point sets are considered and no structural information is revealed has been given in

[13]. For our purposes we deploy a "semantically richer" characterization and approximation of fuzzy regions which describes them as *collections of crisp α-level regions*[2] [13]. This view defines a fuzzy region in terms of regularized, nested α-cuts. Let $\tilde{F}$ be a fuzzy region. Then we represent a region F_α for an $\alpha \in [0,1]$ as

$$F_\alpha = reg_c(\{(x,y) \in \mathbb{R}^2 \mid \mu_{\tilde{F}}(x,y) \geq \alpha\})$$

We call F_α an *α-level region.* Clearly, F_α is a crisp complex region whose boundary is defined by all points with membership value α. In particular, F_α can have holes and consist of multiple parts. The kernel of $\tilde{F}$ is then equal to $F_{1.0}$. An essential property of the α-level regions of a fuzzy region is that they are nested, i.e., if we select membership values $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} = 0$ for some $n \in \mathbb{N}$, then $F_{\alpha_1} \subseteq F_{\alpha_2} \subseteq \cdots \subseteq F_{\alpha_n} \subseteq F_{\alpha_{n+1}}$. We here describe the finite case. If $\Lambda_{\tilde{F}}$ is infinite, then there are obviously infinitely many α-level regions which can only be finitely represented within this view if we make a finite selection of α-values. In the finite case, if $|\Lambda_{\tilde{F}}| = n+1$ and if we take all these occurring membership values of a fuzzy region, we can even replace "$\subseteq$" by "$\subset$" in the inclusion relationships above. This follows from the fact that for any $p \in F_{\alpha_i} - F_{\alpha_{i-1}}$ with $i \in \{2, \ldots, n+1\}$, $\mu_{\tilde{F}}(p) = \alpha_i$. For the continuous case, we get $\mu_{\tilde{F}}(p) \in [\alpha_i, \alpha_{i-1})$. As a result, we obtain:

> A *fuzzy region* is a (possibly infinite) set of α-level regions, i.e., $\tilde{F} = \{F_{\alpha_i} \mid 1 \leq i \leq |\Lambda_{\tilde{F}}|\}$ with $\alpha_i > \alpha_{i+1} \Rightarrow F_{\alpha_i} \subseteq F_{\alpha_{i+1}}$ for $1 \leq i \leq |\Lambda_{\tilde{F}}| - 1$.

In Section 5 we will use this characterization for a definition of topological predicates on fuzzy regions. We can then reduce these predicates to topological predicates on collections of crisp regions. Unfortunately, the 9-intersection model only provides topological predicates for simple regions. Hence, we first need to generalize this concept to topological predicates for complex crisp regions. An essential requirement of such a collection is that any two predicates are mutually exclusive and that all predicates together cover all topological configurations.

4 Topological Predicates on Complex Crisp Regions

It is not an objective of this paper to find *all* possible topological relationships between two complex regions. We here confine ourselves to a straightforward generalization of the eight topological relationships for simple regions to complex regions. This procedure may be regarded as an ad hoc approach leading to too coarse predicates. But for many spatial applications this predicate collection is practicable enough, and a more fine-grained differentiation is even not desired.

In the following we use as a syntactical simplification the notation $(P_1|P_2|\ldots|P_n)(F,G)$ for the term $P_1(F,G) \vee P_2(F,G) \vee \ldots \vee P_n(F,G)$ where $P_i : region \times region \rightarrow \mathbb{B}$ is a topological predicate for each $1 \leq i \leq n$.

[2] Other structured characterizations given in [13] describe fuzzy regions as multi-component objects, as three-part crisp regions, and as α-partitions.

4.1 Topological Predicates on Simple Regions with Holes

As a first step to a general definition of topological predicates for complex crisp regions we consider such predicates for simple regions with holes and base their definition on the topological predicates for simple regions as they have been derived from the 9-intersection model (Section 2.2).

Let F and G be two simple regions with holes, that is, $F = F_0 - \bigcup_{i=1}^{n} F_i$ and $G = G_0 - \bigcup_{j=1}^{m} G_j$. We consider F and G to be disjoint if they have nothing in common, that is, either F_0 and G_0 are disjoint, and thus implicitly also their corresponding holes due to the definition of F and G, or F_0 (or G_0, respectively) and implicitly its holes are completely inside a hole G_j of G (F_i of F, respectively) (Figure 2a). Formally, we can then define the predicate $\mathit{disjoint}_{srh}$ as

$$\begin{aligned} \mathit{disjoint}_{srh}(F,G) \quad := \quad & \mathit{disjoint}(F_0, G_0) \vee \\ & (\exists 1 \leq i \leq n : \mathit{inside}(G_0, F_i)) \vee \\ & (\exists 1 \leq j \leq m : \mathit{inside}(F_0, G_j)) \end{aligned}$$

The predicate meet_{srh} is defined as follows (Figure 2b):

$$\begin{aligned} \mathit{meet}_{srh}(F,G) \quad := \quad & \mathit{meet}(F_0, G_0) \vee \\ & (\exists 1 \leq i \leq n : \mathit{coveredBy}(G_0, F_i)) \vee \\ & (\exists 1 \leq j \leq m : \mathit{coveredBy}(F_0, G_j)) \end{aligned}$$

We consider F to be inside G if F_0 is inside G_0 and if each hole G_j of G is either disjoint from F_0 or inside a hole F_i of F. (Figure 3a). The definition for the predicate inside_{srh} is:

$$\begin{aligned} \mathit{inside}_{srh}(F,G) \quad := \quad & \mathit{inside}(F_0, G_0) \wedge \\ & (\forall 1 \leq j \leq m : \mathit{disjoint}(F_0, G_j) \vee \\ & \quad (\mathit{inside}(G_j, F_0) \ \wedge \ \exists 1 \leq i \leq n : \mathit{inside}(G_j, F_i))) \end{aligned}$$

We do not have to take into account the topological relationships between the F_i's and G_0 in our definition, because $\mathit{inside}(F_0, G_0) \Rightarrow \mathit{inside}(F_i, G_0)$ due to $F_i \subset F_0$ for $1 \leq i \leq n$. The predicate $\mathit{contains}_{srh}$ is symmetric to the predicate inside_{srh}, that is, $\mathit{contains}_{srh}(F,G) := \mathit{inside}_{srh}(G,F)$.

We consider F and G to be equal if F_0 and G_0 are equal, if F and G have the same number of holes, and if each hole F_i of F coincides with a hole G_j of G and vice-versa, that is,

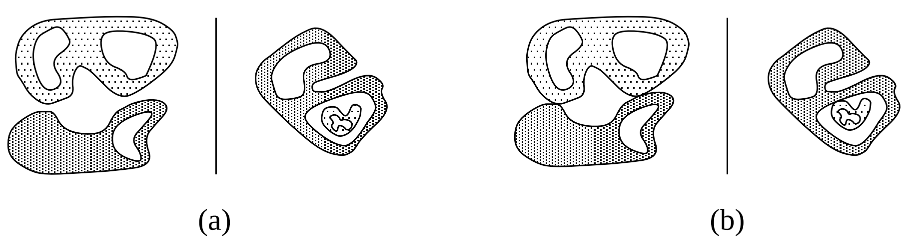


Fig. 2. Examples for the predicates $\mathit{disjoint}_{srh}(F,G)$ (a) and $\mathit{meet}_{srh}(F,G)$ (b).

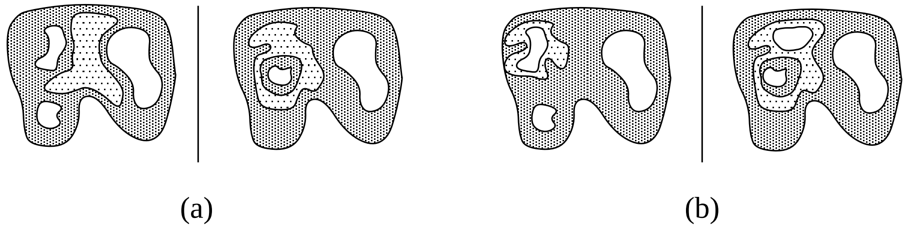


Fig. 3. Examples for the predicates $inside_{srh}(F,G)$ (a) and $coveredBy_{srh}(F,G)$ (b).

$$\begin{aligned} equal_{srh}(F,G) \;:=\; & equal(F_0,G_0) \;\wedge\; n=m \;\wedge \\ & \exists\,\pi:\{1,\ldots,n\}\rightarrow\{1,\ldots,n\}, \pi \text{ bijective}, \\ & \forall\,1\le i\le n: equal(F_i,G_{\pi(i)}) \end{aligned}$$

F is considered to be covered by G if F is a proper subset of G and if F's boundary touches G's boundary (Figure 3b).

$$\begin{aligned} coveredBy_{srh}(F,G) \;:=\; & \neg((inside_{srh}|equal_{srh})(F,G)) \;\wedge \\ & (inside|coveredBy|equal)(F_0,G_0) \;\wedge \\ & (\forall\,1\le j\le m: ((disjoint|meet)(F_0,G_j) \;\vee \\ & (\exists\,1\le i\le n: (inside|coveredBy|equal)(G_j,F_i)))) \end{aligned}$$

The predicate $covers_{srh}$ is symmetric to the predicate $coveredBy_{srh}$, that is, $covers_{srh}(F,G) := coveredBy_{srh}(G,F)$.

Finally, the predicate $overlap_{srh}$ (Figure 4) covers all remaining topological situations. This predicate can, of course, be defined directly in order to give an exact characterization of the remaining topological situations. But this makes the definition unnecessarily complicated and longish. We define instead:

$$\begin{aligned} overlap_{srh}(F,G) \;:=\; & \neg((disjoint_{srh}|meet_{srh}|coveredBy_{srh}|covers_{srh}| \\ & inside_{srh}|contains_{srh}|equal_{srh})(F,G)) \end{aligned}$$

The set $T_{srh} = \{disjoint_{srh},\ meet_{srh},\ overlap_{srh},\ coveredBy_{srh},\ covers_{srh},\ inside_{srh},\ contains_{srh},\ equal_{srh}\}$ provides a complete coverage of topological relationships for two simple regions with holes, and its elements are mutually exclusive. Completeness of T_{srh} follows immediately from the complementary character of the definition of $overlap_{srh}$. Hence, at least one predicate must hold

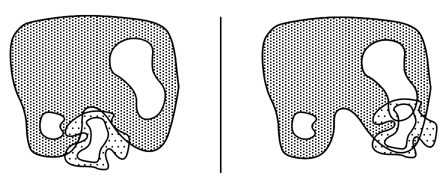


Fig. 4. Examples for the predicate $overlap_{srh}(F,G)$.

for any pair F, G of simple regions with holes. Mutual exclusion of each pair of different topological predicates P_1 and P_2 can be proved by showing that $\neg(P_1(F,G) \ \wedge \ P_2(F,G))$ holds for any pair F, G. Since we have $k = 8$ predicates, we have to check the diversity of $\frac{1}{2}(k^2 - k) = 28$ predicate pairs. We will not show the validity for the predicate pairs here in detail but only tell the strategy. First, we can use the mutual exclusion of the topological relationships for simple regions employed in the definition of some predicates. For instance, $disjoint_{srh}$ and $meet_{srh}$ exclude each other since *disjoint* and *meet* as well as *inside* and *coveredBy* are mutually exclusive in the 9-intersection model. Second, several predicates use the negation of other predicates on simple regions with holes for their definition. For instance, $overlap_{srh}$ excludes all other predicates. Similarly, $coveredBy_{srh}$ excludes both $inside_{srh}$ and $equal_{srh}$. Overall, at most one predicate is valid for any pair F, G.

The set T_{srh} of topological predicates on simple regions with holes is in two ways compatible with the set T of topological predicates on simple regions obtained by the 9-intersection model. First, if both F and G do not have holes, then T_{srh} and T coincide. Second, each of the eight topological predicates on simple regions with holes has the same boolean results for the nine intersections as the corresponding predicate on simple regions (see Section 2).

4.2 Topological Predicates on Complex Regions

With the aid of the topological predicates on simple regions with holes we are now able to define the corresponding predicates on complex regions. Let $F = \bigcup_{i=1}^{n} F_i$ and $G = \bigcup_{j=1}^{m} G_j$ be complex regions where the F_i and G_j are simple regions possibly with holes. We define the following predicates:

$$
\begin{aligned}
disjoint_{cr}(F,G) &:= \forall\, 1 \le i \le n\ \forall\, 1 \le j \le m : disjoint_{srh}(F_i, G_j)\\
meet_{cr}(F,G) &:= \neg disjoint_{cr}(F,G) \ \wedge\\
&\quad\ (\forall\, 1 \le i \le n\ \forall\, 1 \le j \le m : (disjoint_{srh} | meet_{srh})(F_i, G_j))\\
inside_{cr}(F,G) &:= \forall\, 1 \le i \le n\ \exists\, 1 \le j \le m : inside_{srh}(F_i, G_j)\\
contains_{cr}(F,G) &:= inside_{cr}(G,F)\\
equal_{cr}(F,G) &:= n = m \ \wedge\ (\exists\, \pi : \{1, \dots, n\} \to \{1, \dots, n\}, \pi \text{ bijective},\\
&\quad\ \forall\, 1 \le i \le n : equal_{srh}(F_i, G_{\pi(i)}))\\
coveredBy_{cr}(F,G) &:= \neg((inside_{cr} | equal_{cr})(F,G)) \ \wedge\\
&\quad\ (\forall\, 1 \le i \le n\ \exists\, 1 \le j \le m :\\
&\qquad\qquad (inside_{srh} | coveredBy_{srh} | equal_{srh})(F_i, G_j))\\
covers_{cr}(F,G) &:= coveredBy_{cr}(G,F)\\
overlap_{cr}(F,G) &:= \neg((disjoint_{cr} | meet_{cr} | coveredBy_{cr} | covers_{cr} |\\
&\qquad inside_{cr} | contains_{cr} | equal_{cr})(F,G))
\end{aligned}
$$

With similar arguments as in the last section we can recognize that two complex regions satisfy exactly one of these topological predicates. In other words, the topological predicates of the set $T_{cr} = \{disjoint_{cr}, meet_{cr}, inside_{cr}, contains_{cr}, equal_{cr}, coveredBy_{cr}, covers_{cr}, overlap_{cr}\}$ are mutually exclusive and complete. Note that the predicates $disjoint_{cr}$, $meet_{cr}$, $equal_{cr}$, and $overlap_{cr}$ are symmetric whereas the others are not.

One could possibly get the impression that in practice most topological configurations of two complex regions will be classified as overlapping. But this is a fallacy. In many geographic applications *spatial partitions* (maps) form the basic underlying structure. Their essential feature is a non-overlapping constraint imposed on the regions composing a partition.

5 Topological Predicates on Fuzzy Regions

In this section we introduce a concept of topological predicates for fuzzy regions. In a similar way as we can generalize the characteristic function $\chi_A : X \rightarrow \{0,1\}$ to the membership function $\mu_{\tilde{A}} : X \rightarrow [0,1]$ (Section 3.2)[3], we can generalize a (binary) predicate $p_c : X \times Y \rightarrow \{0,1\}$ to a (binary) *fuzzy predicate* $p_f : \tilde{X} \times \tilde{Y} \rightarrow [0,1]$. Hence, the value of a fuzzy predicate can be interpreted as the degree to which the predicate holds for its operand objects. In our case of topological predicates, $X = Y =$ *region*, $\{0,1\} =$ *bool*, and $\tilde{X} = \tilde{Y} =$ *fregion* hold. For the set $[0,1]$ we have to introduce a new type *fbool* for *fuzzy booleans.*

For the definition of fuzzy topological predicates, we take the view of a fuzzy region as a set of α-level regions (Section 3.3). We know that an α-level region is a crisp complex region (Section 3.1), and in the last section we have defined topological predicates on complex regions. This preparatory work now enables us to reduce topological predicates on fuzzy regions to topological predicates on collections of crisp regions[4].

The approach presented in this section is generic in the sense that any meaningful collection of topological predicates on complex crisp regions can be the basis for our definition of a collection of topological predicates on complex fuzzy regions. If the former collection additionally fulfils the properties of completeness and mutual exclusion (which is the case for T_{cr}), the latter collection automatically inherits these properties.

The open question now is how to compute the topological relationships of two collections of α-level regions, each collection describing a fuzzy region. We use the concept of basic probability assignment [5] for this purpose. A *basic probability assignment* $m(F_{\alpha_i})$ can be associated with each α-level region F_{α_i} and can be interpreted as the probability that F_{α_i} is the "true" representative of F. It is defined as

$$m(F_{\alpha_i}) = \alpha_i - \alpha_{i+1}$$

---

[3] Note that χ_A is a unary crisp predicate and that $\mu_{\tilde{A}}$ is a unary fuzzy predicate.

[4] Another great benefit of this approach is its easy implementability through well known concepts for crisp spatial objects and for crisp topological predicates.

for $1 \leq i \leq n$ for some $n \in \mathbb{N}$ with $\alpha_1 = 1$ and $\alpha_{n+1} = 0$. That is, m is built from the differences of successive α_i's. It is easy to see that the telescoping sum $\sum_{i=1}^{n} m(F_{\alpha_i}) = \alpha_1 - \alpha_{n+1} = 1 - 0 = 1$.

Let $\pi_f(F, G)$ be the value that represents a (binary) property π_f between two fuzzy regions F and G. Based on the work in [5] property π_f of F and G can be determined as the summation of weighted predicates by[5]

$$\pi_f(F, G) = \sum_{i=1}^{n} \sum_{j=1}^{n} m(F_{\alpha_i}) \cdot m(G_{\alpha_j}) \cdot \pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$$

where $\pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$ yields the value of the corresponding property π_{cr} for two crisp α-level regions F_{α_i} and G_{α_j}. This formula is equivalent to

$$\pi_f(F, G) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1}) \cdot \pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$$

If π_f is a topological predicate of $T_f = \{disjoint_f,\ meet_f,\ overlap_f,\ equal_f,\ inside_f,\ contains_f,\ covers_f,\ coveredBy_f\}$ between two fuzzy regions, we can compute the degree of the corresponding relationship with the aid of the pertaining crisp topological predicate $\pi_{cr} \in T_{cr}$. The value of $\pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$ is either 1 (*true*) or 0 (*false*). Once this value has been determined for all combinations of α-level regions from F and G, the aggregated value of the topological predicate $\pi_f(F, G)$ can be computed as shown above. The more fine-grained the level set Λ for the fuzzy regions F and G is, the more precisely the fuzziness of topological predicates can be determined.

It remains to show that $0 \leq \pi_f(F, G) \leq 1$ holds, that is, π_f is really a fuzzy predicate. Since $\alpha_i - \alpha_{i+1} > 0$ for all $1 \leq i \leq n$ and since $\pi_{cr}(F_{\alpha_i}, G_{\alpha_j}) \geq 0$ for all $1 \leq i, j \leq n$, $\pi_f(F, G) \geq 0$ holds. We can show the other inequality by determining an upper bound for $\pi_f(F, G)$:

$$\begin{aligned}
\pi_f(F, G) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1}) \cdot \pi_{cr}(F_{\alpha_i}, G_{\alpha_j}) \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1}) \qquad (\text{since } \pi_{cr}(F_{\alpha_i}, G_{\alpha_j}) \leq 1) \\
&= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2) + \cdots + (\alpha_1 - \alpha_2)(\alpha_n - \alpha_{n+1}) + \cdots + \\
&\quad\ (\alpha_n - \alpha_{n+1})(\alpha_1 - \alpha_2) + \cdots + (\alpha_n - \alpha_{n+1})(\alpha_n - \alpha_{n+1}) \\
&= (\alpha_1 - \alpha_2)\big((\alpha_1 - \alpha_2) + \cdots + (\alpha_n - \alpha_{n+1})\big) + \cdots + \\
&\quad\ (\alpha_n - \alpha_{n+1})\big((\alpha_1 - \alpha_2) + \cdots + (\alpha_n - \alpha_{n+1})\big) \\
&= (\alpha_1 - \alpha_2) + \cdots + (\alpha_n - \alpha_{n+1}) \qquad (\text{since } \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) = 1) \\
&= 1
\end{aligned}$$

[5] For reasons of simplicity, we assume that $\Lambda_{\tilde{F}} = \Lambda_{\tilde{G}} =: \Lambda$. Otherwise, it is not difficult to "synchronize" $\Lambda_{\tilde{F}}$ and $\Lambda_{\tilde{G}}$ by forming their union and by reordering and renumbering all levels.

Hence, $\pi_f(F,G) \leq 1$ holds.

An alternative definition of fuzzy topological predicates, which pursues a similar strategy like the one discussed so far, is based on the topological predicates on simple regions possibly with holes, that is, on predicates $\pi_{srh} \in T_{srh}$. If F_{α_i} is an α-level region, let us denote its faces by $F_{\alpha_{i1}}, \ldots, F_{\alpha_{if_i}}$. Similarly, we denote the faces of an α-level region G_{α_j} by $G_{\alpha_{j1}}, \ldots, G_{\alpha_{jg_j}}$. We can then define a topological predicate π'_f as

$$\pi'_f(F,G) = \sum_{i=1}^{n}\sum_{k=1}^{f_i}\sum_{j=1}^{n}\sum_{l=1}^{g_j} \frac{(\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1}) \cdot \pi_{srh}(F_{\alpha_{ik}}, G_{\alpha_{jl}})}{f_i \cdot g_j}$$

It is obvious that $\pi'_f(F,G) \geq 0$ holds since all factors have a value greater than or equal to 0. We can also show that $\pi'_f(F,G) \leq 1$ by the following transformations:

$$\begin{aligned}
\pi'_f(F,G) &\leq \sum_{i=1}^{n}\sum_{k=1}^{f_i}\sum_{j=1}^{n}\sum_{l=1}^{g_j} \frac{(\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1})}{f_i \cdot g_j} \qquad (\pi_{srh}(F_{\alpha_{ik}}, G_{\alpha_{jl}}) \leq 1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1})}{f_i \cdot g_j} \cdot f_i \cdot g_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_i - \alpha_{i+1}) \cdot (\alpha_j - \alpha_{j+1}) \\
&= 1
\end{aligned}$$

Hence, $\pi'_f(F,G) \leq 1$ holds. As a rule the predicates π_f and π'_f do not yield the same results. Assume that F_{α_i} and G_{α_j} fulfil a predicate $\pi_{cr} \in T_{cr}$. This fact contributes once to the summation process for π_f. But it does not take into account that possibly several faces $F_{\alpha_{ik}}$ (at least one) of F_{α_i} satisfy the corresponding predicate $\pi_{srh} \in T_{srh}$ with several faces $G_{\alpha_{jl}}$ (at least one) of G_{α_j}. This fact contributes several times (at most $f_i \cdot g_j$) to the summation process for π'_f. Hence, the evaluation process for π'_f is more fine-grained than for π_f.

Both generic predicate definitions reveal their quantitative character. If the predicate $\pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$ and the predicate $\pi_{srh}(F_{\alpha_{ik}}, G_{\alpha_{jl}})$, respectively, is never fulfilled, the predicate $\pi_f(F,G)$ and $\pi'_f(F,G)$, respectively, yields *false*. The more α-level regions of F and G (simple regions with holes of F_{α_i} and G_{α_j}) fulfil the predicate $\pi_{cr}(F_{\alpha_i}, G_{\alpha_j})$ ($\pi_{srh}(F_{\alpha_{ik}}, G_{\alpha_{jl}})$), the more the validity of the predicate π_f (π'_f) increases. The optimum is reached if all topological predicates are satisfied.

6 Conclusions

In this paper we have developed a formal and coherent definition for simple regions with holes, crisp complex regions, fuzzy complex regions, and for corresponding topological predicates. Spatial query langages can now also be employed to pose queries using topological relationships on more complex regions. For fuzzy predicates their computationally determined quantification has to be additionally considered in a query language. A solution could be to embed adequate qualitative linguistic descriptions of topological relationships as appropriate interpretations of the membership values into spatial query languages. For instance, depending on the membership value yielded by the predicate $inside_f$, we could distinguish between *a little bit inside*, *somewhat inside*, *quite inside*, *nearly completely inside*, and *completely inside*. These linguistic terms could then be incorporated into spatial queries. Another subject of further investigation will be how these spatial data types and topological predicates can be implemented in an efficient, numerically robust, and topologically consistent manner.

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