Algorithms for Hierarchical Spatial Reasoning

Dimitris Papadias

Artificial Intelligence Research Division
Institute for Applied Information Technology
GMD
D-53754 Sankt Augustin, Germany
dimitris@nathan.gmd.de
fax: +49-2241-142072
tel: +49-2241-142329

Max J. Egenhofer

National Center for Geographic Information and Analysis and
Department of Spatial Information Science and Engineering,
Department of Computer Science
University of Maine
Orono, ME 04469-5711, USA
max@spatial.maine.edu
fax: 207- 581-2206
tel: 207-581-2127

Abstract In several applications, there is the need to reason about spatial relations using multiple local frames of reference that are organized hierarchically. This paper focuses on hierarchical reasoning about *direction relations*, a special class of spatial relations that describe order in space (e.g., north or northeast). We assume a spatial database of points and regions. Points belong to regions, which may recursively be parts of larger regions. The direction relations between points in the same region are explicitly represented (and not calculated from coordinates). Inference mechanisms are applied to extract direction relations between points located in different regions and to detect inconsistencies. We study two complementary types of inference. The first one derives the direction relation between points from the relations of their ancestor regions. The second type derives the relation through chains of common points using path consistency. We present algorithms for both types of inference and discuss their computational complexity.

Keywords: Spatial Reasoning, Geographic Databases, Direction Relations, Path Consistency

This work was partially supported by the National Science Foundation under grant number No. SES 88-10917 for the NCGIA. Dimitris Papadias is currently supported by an ERCIM fellowship. Max Egenhofer's work is further supported by the National Science Foundation under grant number IRI-9309230, Rome Laboratories under grant number F30602-95-1-0042, the Scientific and Environmental Division of the North Atlantic Treaty Organization, and by a Massive Digital Data Systems contract sponsored by the Advanced Research and Development Committee of the Community Management Staff and administered by the Office of Research and Development.

1. INTRODUCTION

Direction relations constitute a special class of spatial relations that deal with order in space (e.g., *left* or *northeast*). The large availability of spatial data from various sources and in various forms, in combination with progress in spatial databases, and geographic information systems (GISs), created the need to answer queries involving direction (and other spatial) relations. This has motivated a significant amount of research on spatial reasoning (Smith and Park, 1992; Egenhofer and Sharma, 1993), spatial query processing (Clementini et al., 1994; Papadias et al., 1995) and spatial query languages (Roussopoulos et al., 1988; Egenhofer, 1994, Papadias and Sellis, 1995).

A number of *relation-based* systems have been proposed for the representation of direction relations. Chang et al., (1987) designed *2D strings* for iconic indexing in image databases. A 2D string is a pair of one-dimensional strings that represent the symbolic projections of the objects on the x and y axis. Glasgow and Papadias (1992) developed *symbolic arrays*, which are nested array structures that preserve directions relations among the distinct parts of complex spatial entities at different levels. Most previous work, however, has focused on the representation and processing of explicit relations and the proposed systems do not include mechanisms for inference and inconsistency checking. Although some approaches have dealt with hierarchical direction reasoning, these mainly concern specific types of hierarchies captured by certain types of representations (Glasgow, 1994).

This paper studies hierarchical reasoning about direction relations in spatial databases that cannot resort to coordinate-based representations. Such situations are typically found in narratives, trip reports, and scientists' field notes. We assume that there only exists relative information about the objects within a region, and inclusion relations (i.e., the hierarchical structure). The goal is to infer the direction relations between objects located in different regions. Hierarchical inference mechanisms are necessary to complement other qualitative spatial reasoning methods (Sharma et al., 1994).

For this purpose, we assume the existence of multiple hierarchies, that is, a spatial entity (region or point) could belong to more than one regions at the next hierarchical level. Direction relations between objects in the same region are explicitly represented and consistent. Inconsistencies may occur by combining spatial knowledge from different sources and alternative representations such as images, topographic surveys or verbal descriptions (for an extended discussion see Frank, 1992). For instance, consider a database of maps (for which no transformations among their reference systems are available). If a location is northeast of another in the map of region A, and the second location is north of a third one in the map of region B, then the existence of the first location south of the third in region C would yield an inconsistency. Spatial inference mechanisms are essential for explicating relations and enforcing consistency in the database.

The rest of the paper is organized as follows: Section 2 defines direction relations between points and regions, and describes spatial databases preserving directions. An algorithm for the inference of the relation between points using the relations of their ancestor regions is presented in Section 3. Section 4 describes a complementary form of inference that uses chains of common points and achieves path consistency for the whole database. Section 5 studies the complexity of the algorithms and discusses information loss issues. Section 6 proposes extensions for alternative types of direction relations, and Section 7 concludes with comments.

2. DIRECTION RELATIONS

There have been two approaches in defining direction relations (Hernandez, 1994). According to the *cone-shaped* approach, direction relations are defined using angular regions between objects (Peuquet, Ci-Xiang, 1987; Dutta, 1989). Our method is *projection based*, that is, direction relations are defined using projection lines vertical to the coordinate axes (Mukerjee and Joe, 1991; Sistla et al., 1994). For the following discussion, P, P_1, P_2 ... denote points, A, A_1, A_2 ... regions, and X, X_1, X_2 ... objects (points or regions). In this paper we are concerned with spatial DBMSs that store only direction relations between distinct objects and not absolute coordinates.

2.1 Direction Relations Between Points and Regions

We use the notation A_ NW(P_1,P_2) to express that point P_1 is NorthWest of P_2 in region A. There are nine “primitive” direction relations between points if we assume projection-based definitions (Freksa, 1992; Papadias and Sellis, 1994). Figure 1 illustrates these relations depending on the position of a primary point with respect to a reference point (denoted by *) in region A. In addition to the eight relations of Figure 1, there is also SamePosition which means that the points are at the same location.

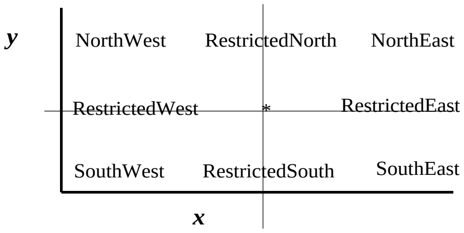


Figure 1 Primitive direction relations between points

Exactly one of the previous relations holds true between any pair of point objects in a region. The primitive relations are transitive and *SP* is also symmetric. The rest form four pairs of converse relations (e.g., A_ NW(P_1,P_2) _ A_ SE(P_2,P_1)). *U* denotes the *universal* relation, the disjunction of all primitive relations. The relation denotes the *empty* relation (the relation that arises during inconsistencies). The above relations form a relation algebra and can be used for relation-based reasoning. They constitute the set of *high resolution* relations; we also define a set of *low resolution* relations using disjunctions:

A_ N(P_1,P_2) ≡ A_ NW(P_1,P_2) A_ RN(P_1,P_2) A_ NE(P_1,P_2) *(North)*

A_ E(P_1,P_2) ≡ A_ NE(P_1,P_2) A_ RE(P_1,P_2) A_ SE(P_1,P_2) *(East)*

A_ S(P_1,P_2) ≡ A_ SW(P_1,P_2) A_ RS(P_1,P_2) A_ SE(P_1,P_2) *(South)*

A_ W(P_1,P_2) ≡ A_ NW(P_1,P_2) A_ RW(P_1,P_2) A_ SW(P_1,P_2) *(West)*

A_ SL(P_1,P_2) ≡ A_ RW(P_1,P_2) A_ SP(P_1,P_2) _ A_ RE(P_1,P_2) *(SameLevel)*

A_ SH(P_1,P_2) ≡ A_ RN(P_1,P_2) A_ SP(P_1,P_2)_ A_ RS(P_1,P_2) *(SamewidtH)*

In order to define direction relations between regions we use projections on the x and y axis. There are 13 mutually exclusive relations between intervals in 1D space (Allen, 1983). If we extend Allen's relations to 2D space we get the 169 primitive relations between region projections of Figure 2. A_ P_{1-1} (A_1,A_2) means that A_1 and A_2 are related by projection relation P_{1-1} in region A that contains A_1 and A_2. Previous structures aimed at the representation of direction relations, (e.g., 2D Strings, Symbolic Arrays) preserve only the above type of relations and discard other forms of spatial information, such as shape, distance and topological relations.

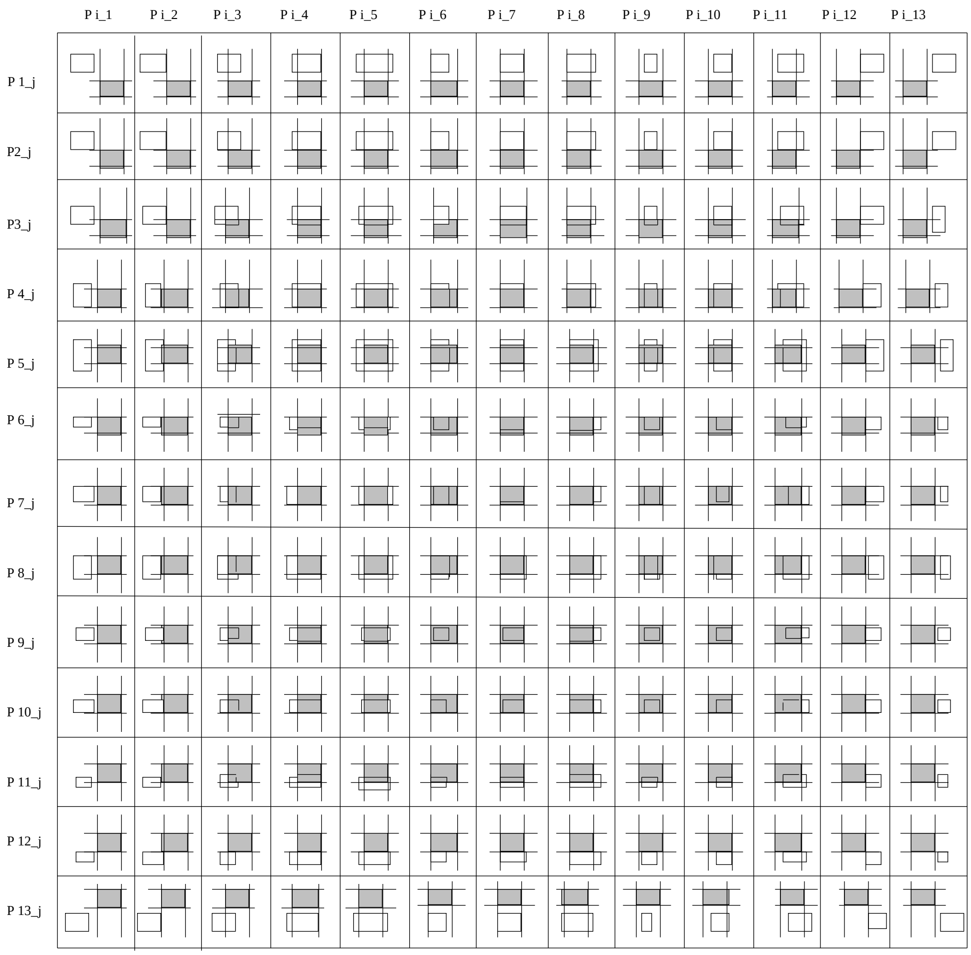


Figure 2 Projection relations in 2D space

2.2 Retrieval of Direction Relations in Spatial Databases

Let *DB* be a spatial database of maps each corresponding to a distinct region. For every map there is a relation-based representation (2D string, symbolic array, a relational table or a set of binary predicates) that stores the relations between all pairs of objects in the region. The objects in the map can be either points or regions but not both (the regions that contain points are called *leaf regions*). Each pair of objects in a map is related by a primitive direction relation explicitly represented. The relation of each point with itself is SP and the projection relation of each region with itself is $P_{7\text{-}7}$. Converse pairs of objects are related by converse relations.

The hierarchy is represented by pointers to next-level areas (IN relation). *DB* _ IN(X_i,A_j) denotes that object (point or region) X_i is a part of (therefore, totally contained in) the next level region A_j. IN* is the transitive closure of IN: *DB* _ IN*(X_i,A_j) _ *DB* _ IN(X_i,A_j) A_k [*DB* _ IN*(X_i,A_k)   *DB* _ IN*(A_k,A_j)]. IN* needs not be explicitly represented, but can be computed by a recursive function that traverses the hierarchy bottom-up, and marks all the ancestors of an object in the hierarchy. For demonstration, we use the example of Figure 3a; Figure 3b illustrates the hierarchy and the relations explicitly represented.

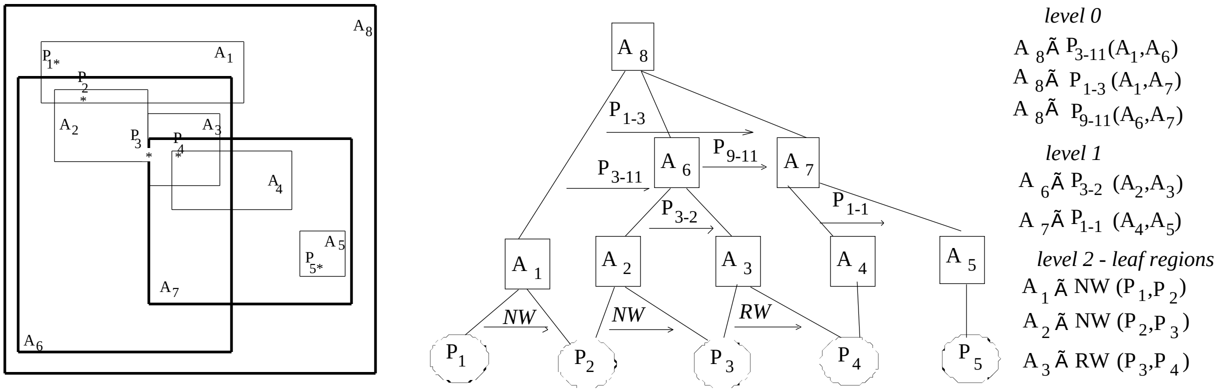


Figure 3 Example of inference through regions

R, R_1, R_2 ... denote relation variables between points, and r, r_1, r_2 ... between regions. The general problem is to retrieve the (explicit, or implicit) relation between any pair of points P_i and P_j in the database: *DB* _ R(P_i,P_j). There are three cases regarding the direction relations between points. The first case is *explicit retrieval,* that is, there is a region A which contains P_i and P_j: A (A _ R(P_i,P_j)) *DB* _ R(P_i,P_j). Inconsistencies during explicit retrieval arise when P_i and P_j exist together in multiple maps and their relations in these maps are different (an inconsistency of this form would be: A_1_ NW(P_1,P_2) and A_2_ NE(P_1,P_2)). The following algorithm performs explicit retrieval by retrieving all leaf regions and examining the relations between all pairs of points in them[1]. We assume an *initialization* process that assigns *U* to the relation between each pair of distinct points (*SP* for identical points).

```
Explicit_retrieval
for each (leaf) region $A_k$
        retrieve $A_k$;
        for each point $P_i$ such that DB_ IN($P_i$,$A_k$)
                        for each point $P_j$ such that DB_ IN($P_j$,$A_k$) and i<j
                                get the relation R′ : $A_k$ _ R′($P_i$,$P_j$);
                                R($P_i$,$P_j$)= R($P_i$,$P_j$)∩R′;
                                if R=   then return INCONSISTENCY DUE TO EXPLICIT RETRIEVAL;
                                        else R($P_j$,$P_i$)=converse(R($P_i$,$P_j$));
                        end-for
        end-for
end-for
```

In the second case, *inference through regions,* P_i and P_j do not exist in the same region, but their relation can be inferred from the relations between their ancestor regions. The notation r→R means that when the relation r holds true between two regions, then the relation R holds between all pairs of points in the regions. For example, $P_{1\text{-}1}$→ NW; if two regions are related by projection relation $P_{1\text{-}1}$, the relation between any two points, each belonging to one region, is NW. Inference through regions can be described as: [A_k A_l (*DB*_ IN*(P_i,A_k) _ *DB* _ IN*(P_j,A_l) _ *DB* _ r(A_k,A_l)) (r → R)] *DB* _ R(P_i,P_j). Inconsistencies during inference through regions arise when the relation between P_i and P_j in some map is not consistent with the relation between some of their ancestors regions. As an example consider: A_ RN(P_1,P_2) and DB_ IN*(P_1,A_1) and DB_ IN*(P_2,A_2) and DB_ $P_{1\text{-}1}$(A_1,A_2).

[1] All the algorithms assume that information in each region is arc consistent: A_R(P_i, P_j) ⇔ A_converse(R(P_j, P_i)) and work only on the pairs (P, P) for which i<j

In the third case (*inference through points*), the relation between P_i and P_j that belong to different maps is inferred by a chain of common points using composition[2] of spatial relations: [P (*DB*_ $R_k(P_i,P)$ _ *DB* _ $R_l(P, P_j)$) (R_k R_l = R)] *DB* _ $R(P_i,P_j)$. Inconsistencies in this case arise when different relations are inferred by different chains of points, or when the inferred relation contradicts the results of explicit retrieval or inference through regions (e.g., A_1_ NW(P_1,P), A_2_ NW(P,P_2) and A_3_ RS(P_1,P_2)).

Unlike explicit retrieval which is straightforward, the other two cases require inference mechanisms that potentially search large parts of the database. In the next sections we discuss algorithms that extract the relation between all pairs of points and detect inconsistencies. For each case we provide rules of inference, describe extensive examples, and obtain formulas for the cost.

3. SPATIAL INFERENCE THROUGH REGIONS

This type of inference usually provides good results with minimum computational overhead, under the condition that the points belong to "ancestor" regions with non-overlapping projections on at least one axis. The same result is not always achievable using inference through common points (even if such common points exist).

3.1 Rules of Inference

In the case that the projections of two regions are disjoint on both axes (projections P_{1-1}, P_{1-13}, P_{13-13}, and P_{13-1} in Figure 2), then high resolution information can be inferred for both south-north and west-east directions. However, not all projections allow such inferences regarding the relations between points. When the projections are disjoint on only one axis, low resolution relations about this axis can be derived, but information on the other axis is lost. Figure 4 summarises the relations that can be derived about points given the projection relation between regions.

P	1	2	3	4	5	6	7	8	9	10	11	12	13
1	NW	NW RN	N	N	N	N	N	N	N	N	N	NE RN	NE
2	NW RW	NW RN RW SP	N SL	N SL	N SL	N SL	N SL	N SL	N SL	N SL	N SL	NE RN RE SP	NE RE
3	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
4	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
5	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
6	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
7	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
8	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
9	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
10	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
11	W	W SH	U	U	U	U	U	U	U	U	U	E SH	E
12	SW RW	SW RS RW SP	S SL	S SL	S SL	S SL	S SL	S SL	S SL	S SL	S SL	SE RS RE SP	SE RE
13	SW	SW RS	S	S	S	S	S	S	S	S	S	SE RS	SE

Figure 4 Direction relation between points implied by the relation of ancestor regions (r → R)

The entries with *U*, correspond to overlapping projections on both axes (in this case there is *information loss* because no conclusion can be drawn about the relations between points). In general, information loss increases as a function of the data *density* (sum of all region areas divided by the area of the global space). However, experiments with spatial access methods for Geographic Information Systems have shown that for usual values of density the vast majority of projections are disjoint (Papadias and Theodoridis, to appear). For verification, we created 10,000 regions of various sizes, randomly distributed over the global space. The percentage of regions that have disjoint projections on both axes with respect to a reference region object varied from 99.9% to 99.5%. Similar numbers are produced by real geographic data sets used as standard benchmarks for databases (see Faloutsos and Kamel, 1994).

The above observations refer to "flat" representations; the hierarchical organization in multiple levels results in increased information loss (for a discussion see Section 5). Nevertheless,

[2] The problem of composition can be defined as "if the spatial relation between P_i and P, and between P and P_j is known, what are the possible relations between P_i and P_j?". The symbol * denotes *path composition* (Frank, 1992): R_1 R_2 = R means that (R_1(X,Z) R_2(Z,Y)) R(X,Y)

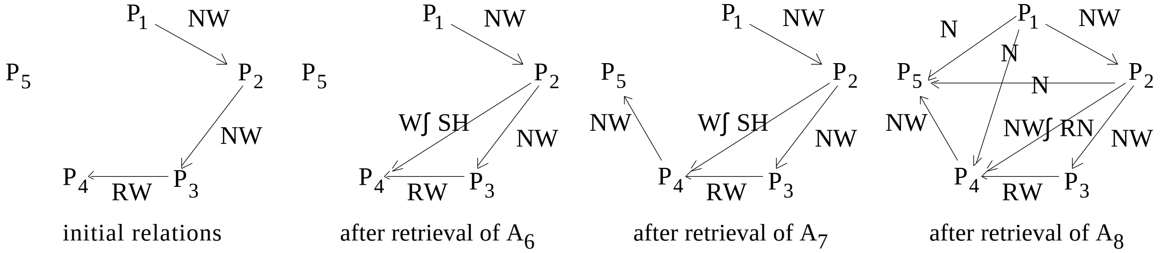
chances are that inference through regions will produce a high resolution relation. Traditional spatial data structures, such as the R-trees (Guttman, 1984), take advantage of this fact for the efficient retrieval of overlap queries by hierarchical decomposition of space. However, spatial data structures assume that global coordinates are stored, and cannot be used when only the relative positions of objects in the same spatial entity are known (as happens here).

3.2 The Algorithm

Initially the relation between any pair of points is given by explicit retrieval. Inference through regions retrieves one by one all non-leaf regions A and gets the relation r: A√ r(A_k,A_l) for all pairs of regions IN A. Let R' be the relation implied by r: r→R' (according to the rules of Figure 4). If R' ≠ U, the relation between all pairs of points P_i such that DB_IN*(P_i,A_k), and P_j such that DB_IN*(P_j,A_l) is updated to R(P_i,P_j)=R(P_i,P_j)∩R'.

```
Inference_through_regions
for each non-leaf region A
  retrieve A;
  for each region $A_k$ such that DB_IN($A_k$,A)
     for each region $A_l$ such that DB_IN($A_l$,A) and k<l
       get the relation r : A _ r($A_k$,$A_l$);
       lookup R’: r → R’;
       if R’≠ U then
           for each point $P_i$ such that DB_IN*($P_i$,$A_k$)
             for each point $P_j$ such that DB_IN*($P_j$,$A_l$)
              R($P_i$,$P_j$)=R($P_i$,$P_j$)∩R’ ;
              if R($P_i$,$P_j$)=  then return INCONSISTENCY DUE TO INFERENCE THROUGH REGIONS;
                             else R($P_j$,$P_i$)=converse(R($P_i$,$P_j$));
             end-for
           end-for
     end-for
  end-for
end-for
```

For demonstration of the algorithm we use the example of Figure 3. Figure 5 illustrates the explicit relations between all pairs of points in the form of a constraint network. First A_6 is retrieved and the relation between A_2 and A_3 is found to be $P_{3\text{-}2}$. Since $P_{3\text{-}2}$→W SH, the relation between P_2 (that belongs to A_2) and P_4 (that belongs to A_3) is refined to U∩(W SH)= W SH (Figure 5b). The relation between P_3 and the other points of A_2 and A_3 remains unchanged because NW∩(W SH)=NW and RW∩(W SH)=RW (for (P_2,P_3) and (P_3,P_4) respectively). Then A_7 is retrieved and the relation NW between P_4 and P_5 is inferred because the ancestor regions of the two points (A_4 and A_5) are related by $P_{1\text{-}1}$ and, $P_{1\text{-}1}$ → NW (Figure 5c). After the retrieval of A_8 (the last non-leaf region) the network takes its final form of Figure 5d. From A_8_ $P_{1\text{-}3}$(A_1,A_7), and $P_{1\text{-}3}$→N, the relation North is inferred between all points of A_1 and the ones IN* A_7, resulting in N(P_1,P_4), N(P_1,P_5), N(P_2,P_5) and NW(P_2,P_4) RN(P_2,P_4) (the last relation is obtained by N∩(W SH)). The relations $P_{3\text{-}11}$(A_1,A_6) and $P_{9\text{-}11}$(A_6,A_7) do not allow any inferences because $P_{3\text{-}11}$ → U and $P_{9\text{-}11}$ → U.



initial relations | after retrieval of A_6 | after retrieval of A_7 | after retrieval of A_8

Figure 5 Illustration of the algorithm

Since the algorithm generates the permitted relations for all pairs of points, it needs to be performed only once and its results can be stored for future use. The above algorithm produces fast (an analysis is given later in the paper) and high resolution relations in many situations. However, in cases where we have overlapping projections with multiple common points (as in Figure 3) further refinements are possible by using the common points.

4. SPATIAL INFERENCE THROUGH POINTS

Inference using common points, can be formulated as a path consistency problem in a network of binary direction constraints. Each constraint in the network is a disjunction of primitive relations and represents the permitted relations between a pair of points after explicit retrieval and inference through regions have taken place (e.g., Figure 5d). Path consistency uses the relative positions of common points to derive the relation between any two points as they are implied by the given constraints. Inference is achieved by excluding relations that cause inconsistencies and maintaining only the ones that could participate in a solution of the network.

4.1 Rules of Inference

In order to apply some path consistency algorithm we need a set of composition rules for direction relations. Figure 6 describes the rules that are applied in order to produce the possible direction relations between P_i and P_j when their relation with respect to a third point P is known. Frank (in press) describes composition of direction relations based on the concepts of *projections* and *cone-shaped directions*. Unlike Frank who uses the notion of *Euclidean approximate* to deal with uncertainty, our system generates a disjunction of the potential primitive relations (which are expressed by the low resolution relations).

	NW (P,P_j)	RN (P,P_j)	NE (P,P_j)	RW (P,P_j)	SP (P,P_j)	RE (P,P_j)	SW (P,P_j)	RS (P,P_j)	SE (P,P_j)	N (P,P_j)	E (P,P_j)	S (P,P_j)	W (P,P_j)	SL (P,P_j)	SH (P,P_j)	U (P,P_j)
NW(P_i,P)	NW	NW	N	NW	NW	N	W	W	U	N	U	U	W	N	W	U
RN(P_i,P)	NW	RN	NE	NW	RN	NE	W	SH	E	N	E	U	W	N	SH	U
NE(P_i,P)	N	NE	NE	N	NE	NE	U	E	E	N	E	U	U	N	E	U
RW(P_i,P)	NW	NW	N	RW	RW	SL	SW	SW	S	N	U	S	W	SL	W	U
SP(P_i,P)	NW	RN	NE	RW	SP	RE	SW	RS	SE	N	E	S	W	SL	SH	U
RE(P_i,P)	N	NE	NE	SL	RE	RE	S	SE	SE	N	E	S	U	SL	E	U
SW(P_i,P)	W	W	U	SW	SW	S	SW	SW	S	U	U	S	W	S	W	U
RS(P_i,P)	W	SH	E	SW	RS	SE	SW	RS	SE	U	E	S	W	S	SH	U
SE(P_i,P)	U	E	E	S	SE	SE	S	SE	SE	U	E	S	U	S	E	U
N(P_i,P)	N	N	N	N	N	N	U	U	U	N	U	U	U	N	U	U
E(P_i,P)	U	E	E	U	E	E	U	E	E	U	E	U	U	U	E	U
S(P_i,P)	U	U	U	S	S	S	S	S	S	U	U	S	U	S	U	U
W(P_i,P)	W	W	U	W	W	U	W	W	U	U	U	U	W	U	W	U
SL(P_i,P)	N	N	N	SL	SL	SL	S	S	S	N	U	S	U	SL	U	U
SH(P_i,P)	W	U	E	W	SH	E	W	SH	E	U	E	U	W	U	SH	U
U(P_i,P)	U	U	U	U	U	U	U	U	U	U	U	U	U	U	U	U

Figure 6 Composition table for low and high resolution relations

The composition constraint R_k * R_l is computed by forming the cross products of the primitive relations that comprise R_k and R_l, composing each resulting ordered pair by looking up the results in the composition table, and taking the union of the resulting sets. Besides the primitive relations, the table of Figure 6 contains the low resolution relations and U. Notice that the set of low and high resolution relations (plus) is closed under composition and intersection.

In addition to the above 16 relations, the type of networks that result after inference through regions, may contain another 16 relations such as NW RN (see Figure 4). The set of 32 (+) relations is also closed under composition and intersection[3]. Figure 7 illustrates the compositions of the five new relations at the upper-left corner of the table of Figure 4 with all relations. The remaining relations of Figure 7 produce the symmetrical relations on the corresponding axes. Therefore, the constraint between each pair of points in the network is always one of the 32

[3]

relations (and   when there is an inconsistency), and arbitrary disjunctions do not appear at any phase of inference through points. As we discuss later, this fact is important for the consistency of the relations in the final network, and for the cost of execution.

		NW(P,P_j)
		RN(P,P_j)
			NW(P,P_j)
			RW(P,P_j)
				NW(P,P_j) RN(P,P_j)
				RW(P,P_j) SP(P,P_j)
					N(P,P_j)
					SL(P,P_j)
						W(P,P_j)
						SH(P,P_j)

1	NW(P_i,P)	NW	NW	NW	N	W
2	RN(P_i,P)	NW RN	NW	NW RN	N	W SH
3	NE(P_i,P)	N	N	N	N	U
4	RW(P_i,P)	NW	NW RW	NW RW	N SL	W
5	SP(P_i,P)	NW RN	NW RW	NW RN RW SP	N SL	W SH
6	RE(P_i,P)	N	N SL	N SL	N SL	U
7	SW(P_i,P)	W	W	W	U	W
8	RS(P_i,P)	W SH	W	W SH	U	W SH
9	SE(P_i,P)	U	U	U	U	U
10	N(P_i,P)	N	N	N	N	U
11	E(P_i,P)	U	U	U	U	U
12	S(P_i,P)	U	U	U	U	U
13	W(P_i,P)	W	W	W	U	W
14	SL(P_i,P)	N	N SL	N SL	N SL	U
15	SH(P_i,P)	W SH	W	W SH	U	W SH
16	NW(P_i,P) RN(P_i,P)	NW RN	NW	NW RN	N	W SH
17	NW(P_i,P) RW(P_i,P)	NW	NW RW	NW RW	N SL	W
18	NW(P_i,P) RN(P_i,P) RW(P_i,P) SP(P_i,P)	NW RN	NW RW	NW RN RW SP	N SL	W
19	N(P_i,P) SL(P_i,P)	N	N SL	N SL	N SL	U
20	W(P_i,P) SH(P_i,P)	W SH	W	W SH	U	W SH
21	SW(P_i,P) RW(P_i,P)	W	W	W	U	W
22	SW(P_i,P) RS(P_i,P)	W SH	W	W SH	U	W SH
23	SW(P_i,P) RS(P_i,P) RW(P_i,P) SP(P_i,P)	W SH	W	W SH	U	W
24	S(P_i,P) SL(P_i,P)	U	U	U	U	U
25	SE(P_i,P) RS(P_i,P)	U	U	U	U	U
26	SE(P_i,P) RE(P_i,P)	U	U	U	U	U
27	SE(P_i,P) RS(P_i,P) RE(P_i,P) SP(P_i,P)	U	U	U	U	U
28	E(P_i,P) SH(P_i,P)	U	U	U	U	U
29	NE(P_i,P) RE(P_i,P)	N	N SL	N SL	N SL	U
30	NE(P_i,P) RN(P_i,P)	N	N	N	N	U
31	NE(P_i,P) RE(P_i,P) RN(P_i,P) SP(P_i,P)	N	N SL	N SL	N SL	U
32	U(P_i,P)	U	U	U	U	U

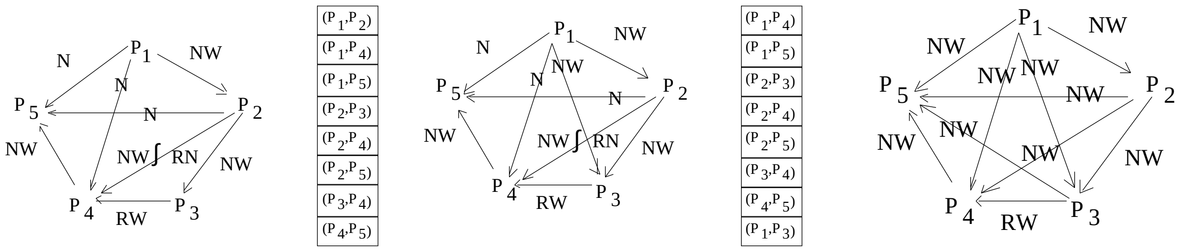
Figure 7 Composition table for relations generated after inference through regions

4.2 The Algorithm

A number of path consistency algorithms have been proposed (Allen, 1983; Macworth and Freuder, 1985). The following one is a variation modified for the current problem. Initially the network is derived from explicit retrieval and inference through regions. All pairs of points whose relation is not U are inserted into a queue. Then every pair is popped from the queue and the corresponding relation is used to refine the relation between the popped points and all the other points that co-exist with them in some region. The pairs of points whose relation is refined are pushed in the queue for propagation of the update through the network.

```
Inference_through_points
for each point $P_i$
       for each point $P_j$ such that $i<j$
                     if $R(P_i,P_j)\neq U$ then push-queue($P_i,P_j$);
while not-empty-queue
    pop-queue($P_i,P_j$);
       for each (leaf) region $A_l$ such that DB_ IN($P_i,A_l$)
              retrieve $A_l$;
              for each point $P_k$ such that DB_ IN($P_k,A_l$) and $k \neq i$ and $k \neq j$
                     $R_t(P_k,P_j)=R(P_k,P_j) \cap (R(P_k,P_i)*R(P_i,P_j))$;
                     if $R_t$=  then return INCONSISTENCY DUE TO PATH CONSISTENCY;
                     else if $R_t(P_k,P_j)$  $R(P_k,P_j)$ then
                                   $R(P_k,P_j)= R_t(P_k,P_j)$;
                                   $R(P_j,P_k)$=converse($R(P_k,P_j)$);
                                   if not in-queue($P_k,P_j$) then push-queue($P_k,P_j$);
              end-for
       end-for
       for each (leaf) region $A_m$ such that DB_ IN($P_j,A_m$)
              retrieve $A_m$;
              for each point $P_k$ such that DB_ IN($P_k,A_m$) and $k \neq i$ and $k \neq j$
                     $R_t(P_i,P_k)=R(P_i,P_k) \cap (R(P_i,P_j)* R(P_j,P_k))$;
                     if $R_t$=  then return INCONSISTENCY DUE TO PATH CONSISTENCY;
                     else if $R_t(P_i,P_k)$  $R(P_i,P_k)$ then
                                   $R(P_i,P_k)= R_t(P_i,P_k)$;
                                   $R(P_k,P_i)$=converse($R(P_i,P_k)$);
                                   if not in-queue($P_i,P_k$) then push-queue($P_i,P_k$);
              end-for
       end-for
end-while
```

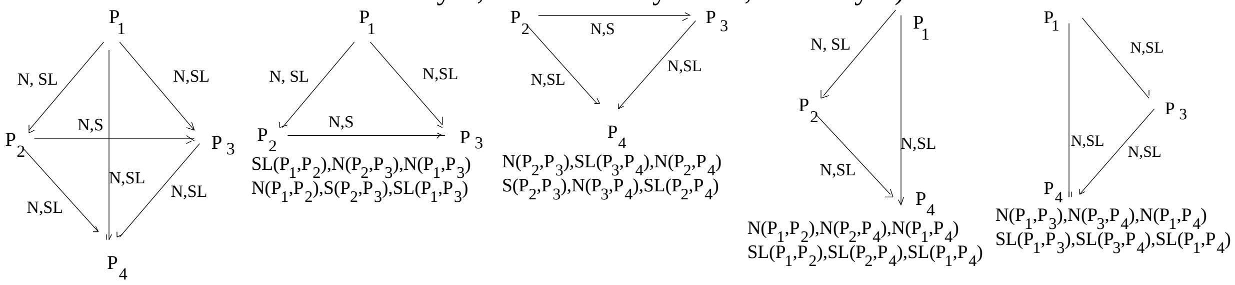
In order to demonstrate the algorithm, we use the configuration of Figure 3 and the network of Figure 5d. After explicit retrieval and inference through regions have been applied, the pairs of points whose relation is not U are pushed into a queue. Here we assume the order of Figure 8a, but the order is not important. First the pair (P_1,P_2) is popped and all the regions that contain these points are retrieved. P_3 co-exists with P_2 in region A_2 and its relation with P_1 is updated according to: $R(P_1,P_3) = R(P_1,P_3) \cap (R(P_1,P_2)*R(P_2,P_3)) = U \cap (NW*NW) = NW$. Because the new relation is a refinement of the previous one (NW U) the pair (P_1,P_3) is pushed into the queue for propagation. The new network and the state of the queue at this phase are illustrated in Figure 8b. Then the pair (P_1,P_4) is popped from the queue, the regions A_1, A_3, and A_4 are retrieved, and the relations between the points (P_2,P_4), and (P_1,P_3) are updated. However the network does not change at this stage because: $R(P_2,P_4) = R(P_2,P_4) \cap (R(P_2,P_1)*R(P_1,P_4)) = (NW\ \ RN) \cap (SE*N) = NW\ \ RN$, and $R(P_1,P_3) = R(P_1,P_3) \cap (R(P_1,P_4)*R(P_4,P_3)) = NW \cap (N*RE) = NW$. Similarly the pair ($P_1$,$P_5$) will not alter the network, while the pair (P_2,P_3) will produce: $R(P_2,P_4)$ =NW. The remaining pairs update the network in the same fashion; the final state after the termination of the algorithm is illustrated in Figure 8c.



initial network and queue after (P_1,P_2) has been popped final network

Figure 8 Illustration of the algorithm

Path consistency refines the constraints between each pair of points by pruning the relations that cause inconsistencies (relations that are not consistent with the explicit relations between some other pairs). However, path consistency does not remove all inconsistencies from general constraint networks[4]. Van Beek and Cohen (1990) have proven that any path consistent, point algebra network that contains inconsistent relations, has a subgraph of four vertices isomorphic to the network of Figure 9a. This network is path consistent because every primitive relation that appears in a constraint participates in at least one solution of each triangle (Figures 9b-9e). Still, the relation SL (*SameLevel*) between P_1 and P_4 causes inconsistency because it will enforce SL between P_2 and P_3, which is not allowed by the initial constraints. Van Beek (1992) demonstrates that such problems are created in networks that contain inequality (N or S but not SL - in our context North can be substituted by >, SameLevel by = and, South by <).



P_1
P_2
P_3
P_4
N, SL
N,SL
N,S
$SL(P_1,P_2),N(P_2,P_3),N(P_1,P_3)$
$N(P_1,P_2),S(P_2,P_3),SL(P_1,P_3)$
$N(P_2,P_3),SL(P_3,P_4),N(P_2,P_4)$
$S(P_2,P_3),N(P_3,P_4),SL(P_2,P_4)$
$N(P_1,P_2),N(P_2,P_4),N(P_1,P_4)$
$SL(P_1,P_2),SL(P_2,P_4),SL(P_1,P_4)$
$N(P_1,P_3),N(P_3,P_4),N(P_1,P_4)$
$SL(P_1,P_3),SL(P_3,P_4),SL(P_1,P_4)$

Figure 9 Path consistent spatial constraint network with inconsistent relations

Nevertheless, in the type of problem we study here, we start with the set of constraints imposed after inference through regions, and not with arbitrary disjunctions. This set is closed under intersection and composition. Therefore, path consistency does not produce inequality (e.g., North South) on any axis, and the network does not contain inconsistent relations after the application of the above algorithm.

5. Complexity and Information Loss Issues

In order to obtain formulas for the cost of the algorithms we make the following simplifications (although such simplifications may not apply for real applications, they provide a good measure for the expected cost in most cases). Each region contains k objects (points or other regions). Each object belongs to m regions in the upper hierarchy level, except for the region at the top (0 level) that does not belong to any region, and the objects at level 1 that belong only to the top-level region. It is always the case that $k/m>1$ and in regular applications $k/m>>1$. N is the total number of points in the database. We assume that there is a buffer that stores the $N(N-1)/2$ relations between all pairs of points.

For demonstration we use Figure 10, where $k=4$ and $m=2$. The objects are represented as nodes in a hierarchy of height h. For each object (except for the ones at levels 0 and 1) there are m copies (illustrated as overlapping nodes), each corresponding to an instance of the object in a parent node (this data replication also exists in the database because each object is represented in all parent regions).

[4] Constraint satisfaction problems are in general exponential in nature, while path consistency is polynomial. Grigni et al., (1995) have proven that constraint satisfaction in networks of topological relations is NP-Complete, and

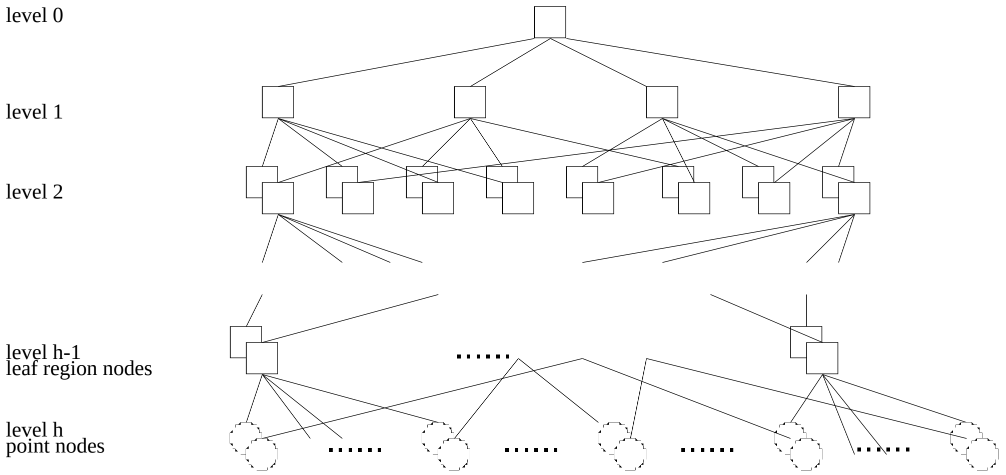


Figure 10 Hierarchical structure for k=4 and m=2

5.1 Cost of Inference

The cost is a function of the *number of map retrievals* because such operations require access to secondary storage (i.e., retrieval of the disk pages that contain the map). This is common practice in database literature where indexing methods are compared on the number of accessed pages from the disk (Guttman 1984; Faloutsos and Kamel, 1994). In the case of explicit retrieval, for example, we have to retrieve all leaf regions. Due to the fact that leaf regions store all points and their copies, their number is *mN/k*, resulting in the same number of map retrievals.

In order to measure the cost of inference through regions we need to calculate the number of non-leaf regions, because all these regions are retrieved. There is only one node at level 0, *k* nodes at level 1, and k^2 at level 2. Out of these k^2 nodes, k^2/m correspond to objects and the rest to copies. Level 3 contains *k* nodes for each original node of the previous level resulting in a total of k^3/m nodes out of which only k^3/m^2 are original and represented at level 4. Similarly, at level h-1 there are k^{h-1}/m^{h-2} nodes that correspond to actual leaf regions. Since the number of leaf regions is *mN/k,* we have the following equation that provides a formula for *h*:

$$\frac{k^{h-1}}{m^{h-2}} = \frac{m?N}{k} ? \quad h = \log_{(k\ /\ m)}\left(\frac{N}{m}\right) \qquad (1)$$

The number of non-leaf regions (and therefore the number of map retrievals during inference through regions) is the sum of original regions from level 0 to level h-2. Substituting the height of equation 1 we get the following approximation for the cost of inference though regions:

$$1 + {}^{h-2}_{i=1} \frac{k^i}{m^{i-1}} \quad m^2 \frac{N-k}{k(k-m)} \qquad (2)$$

In order to find the cost of inference through points we start with the observation that only 32 different constraints may appear in the network. A constraint imposed by inference through regions or explicit retrieval may be refined a number of times until it reaches its final state at the end of path consistency. Each time a refinement happens the corresponding pair of points is pushed to the queue. Figure 11 illustrates the possible refinements for the 32 constraints. A constraint at any level may only be refined to a constraint of a lower level. For example, a constraint between two points may initially be U and become N SL, then NW RN RW SP, then NW RN and finally NW. The links in Figure 11 connect each constraint with the constraints of the immediately lower level that it can be refined. The maximum number of refinements for any constraint is four.

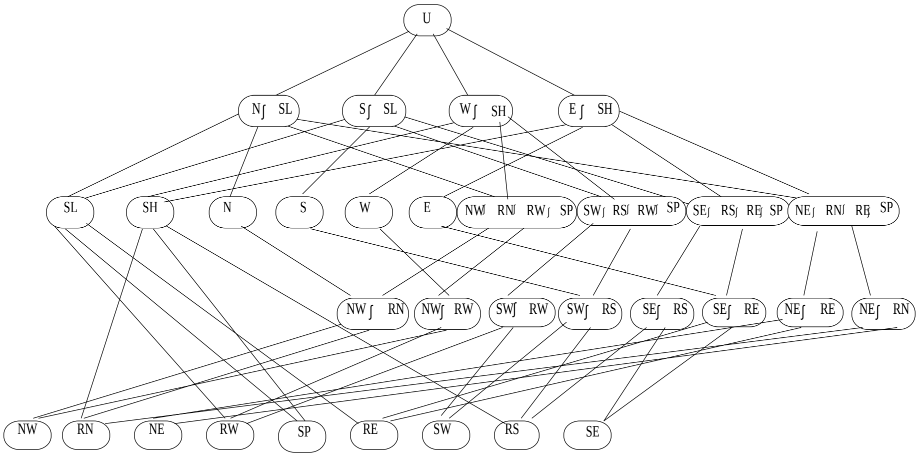


Figure 11 Refinements of direction constraints

There exist *N(N-1)/2* distinct pairs of points in the database and each may be pushed into the queue a maximum of four times. Each time a pair is popped from the queue, *2m* map retrievals are performed to retrieve the points that are related with the popped points in some region. Therefore, inference through points requires *4mN(N-1)* map retrievals in the worst case, which makes it significantly more expensive than inference through regions.

5. 2 Information Loss

The hierarchical decomposition of space in natural geographic entities optimizes queries that involve objects within the same entity ("find all major cities of France northeast of Paris"). For such queries only the map(s) corresponding to the entity (i.e., France) needs to be retrieved, while in a flat representation the same query could involve searching the whole database in the worst case. The trade-off that hierarchical representations pay is information loss regarding the relations between points that exist in different regions, when such relations cannot be inferred.

Consider the example of Figure 12 where P_1 belongs to A_1 which in turn belongs to A_3 and so on. The grey zones show the areas of information loss, that is, areas that correspond to projections of P_1's ancestor at the current level. A high resolution relation between P_1 and some other point can be inferred only if an ancestor of the second point is disjoint with the corresponding grey zones of its level. When a region overlaps one or more of these zones, information about one or both axes is lost. For example, the inferred relation between P_1 and P_2 is South ($P_{13-3} \rightarrow$ S); no east-west information can be extracted because A_2 overlaps the grey area corresponding to A_1's projection on the x axis. Similarly, no relation can be inferred between P_1 and P_3, because A_3 and A_5 have overlapping projections on both axes (although the leaf regions A_1 and A_4 have disjoint projections their relation is not explicitly represented in some parent region). In general, the information loss increases as the distance of the points in the hierarchy increases.

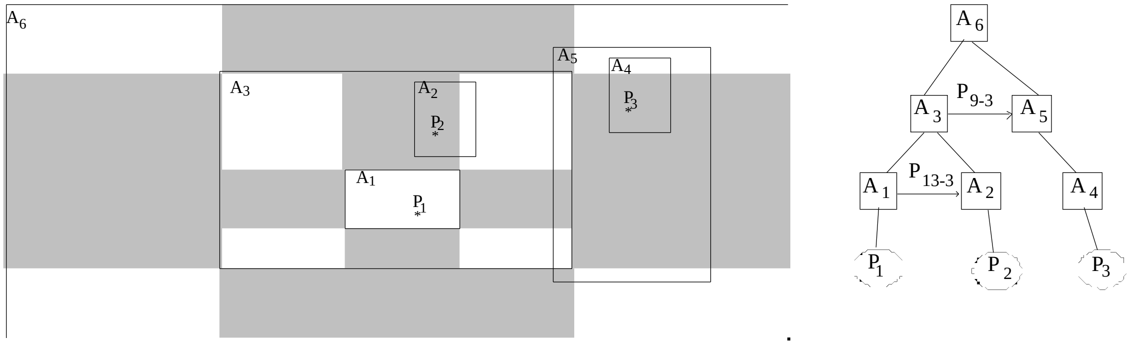


Figure 12 Information loss in hierarchical representations

However, in practical applications this information loss may not be very important because the majority of spatial queries refers to objects within the same geographic entity. Queries of the form "find all cities of France north of Africa" are not common, and if they are imposed, chances are that there is enough information to infer the answer.

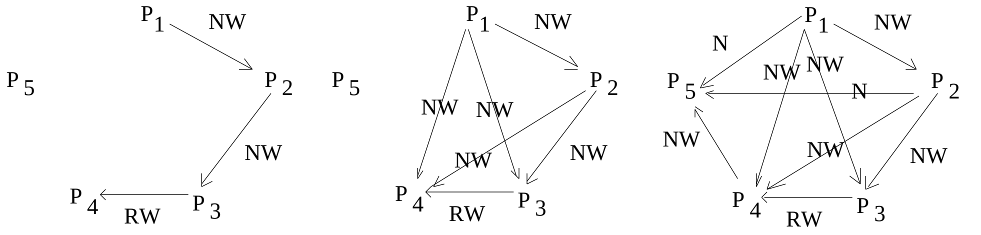
6. DISCUSSION

In this section we describe a unified framework for hierarchical spatial inference, and we discuss its application to alternative forms of direction relations.

6.1 A Unified Framework for Hierarchical Spatial Inference

In the previous sections we argued that first explicit retrieval obtains the relations between pairs of points that exist in the same region, then inference through regions generates additional constraints imposed by the relations between the ancestor regions, and finally inference through points takes advantage of common points to produce further refinements. The order in which explicit retrieval and inference though regions are performed is not important. As long as the content of the database remains unaltered they will generate the same result independently on which is performed first. On the other hand, inference through points has always to be performed at the end, otherwise it may not produce all relations.

Assume, that path consistency is applied before inference through regions to the configuration of Figure 3. The explicit relations are illustrated in Figure 13a, and the path consistent network in Figure 13b. The subsequent application of inference through regions will refine some relations (in particular the relations between P_5 and P_1, P_2, P_4) resulting in the network of Figure 13c which lacks some relations with respect to the network of Figure 8c (e.g., the relation between P_5 and P_3 is *U*). The problem is created by isolated regions (regions, such as A_5, that do not contain common points with other regions). Inference through points has to be applied again in order propagate the new relations and generate the network of Figure 8c.



after explicit retrieval after inference through points after inference through regions

Figure 13 Permutation of the functions

Figure 14 illustrates a unified framework for inference and inconsistency checking in spatial databases of points and objects involving the previous mechanisms. Either explicit retrieval, or inference through regions can be applied after the initialization. Inference through points should be the final phase. This framework explicates the relations between all pairs of points and its results can be stored and used to answer future queries involving direction relations. It should be executed either when there is an update in the database, at periodical time intervals as batch processes, or after the number of modifications in the database becomes larger than a specified threshold.

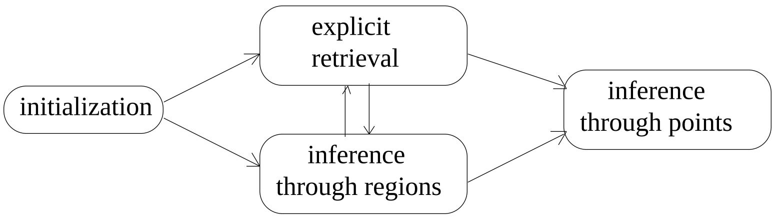


Figure 14 A Unified framework

6.2 Alternative Sets of Direction Relations

Unlike topological relations where there is a set (Egenhofer and Franzosa, 1991) of widely used relations in both research literature and commercial products, there are not universally accepted definitions for direction relations. People have used different types of direction relations to match different needs that range from cognitive modelling (Herskovits, 1986) to image similarity retrieval (Lee et al., 1992) and from robot navigation (Holmes and Jungert, 1992) to user interfaces (Roussopoulos et al., 1988). Although, in this paper, we have assumed a specific set of relations, the algorithms for hierarchical reasoning can be applied to any direction relations with the corresponding inference rules.

As an alternative set, we demonstrate *direction relations with neutral area* first defined in (Frank, 1992). According to these definitions the restricted relations are not line segments, but areas that extend *d/2* from each side of the reference point (where *d* is determined by the application requirements). SamePosition is a square of side d. Figure 15 illustrates the direction relations with neutral area between points. Such relations are useful in cases that there is uncertainty of location.

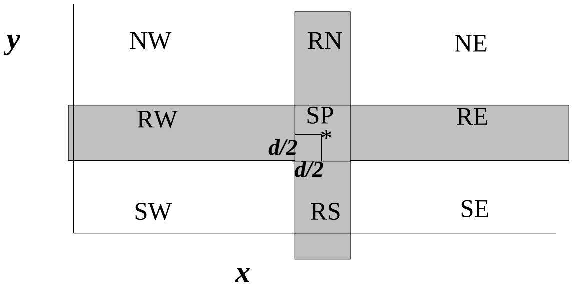


Figure 15 Direction relations with neutral area between points

Projection relations with neutral area can be defined accordingly for regions. There exist 15 (instead of 13) relations on each axis, because there are two new relations for the cases where the primary region is totally contained within *d/2* distance from some edge point. Figure 16 illustrates the first 45 of the 225 possible relations between region projections. Topaloglou (1994) defined similar projections relations to model objects with fuzzy boundaries, an application domain unsuitable for the direction relations of the previous sections.

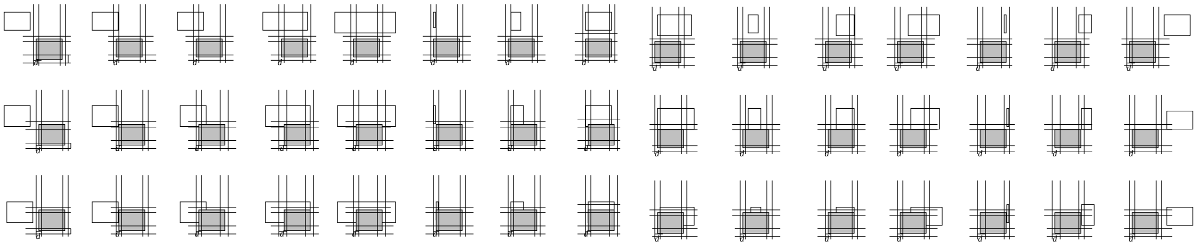


Figure 16 Direction relations with neutral area between regions

Inference through regions can be achieved if we assume the region relations of Section 2 (i.e., relations between points are defined according to Figure 15, and relations between regions according to Figure 2). However, in this case there is information loss even for disjoint projections on both axes. For example, P_{1-1} would imply NW RN RW SP instead of NW. On the other hand, assuming the region relations of Figure 16, the table for inference through regions is identical to the one of Figure 4 (with the inclusion of two extra rows and columns for the additional relations). On the other hand, the composition table for directions with neutral area (Figure 17) is significantly different from the one in Figure 6 (e.g., the restricted relations are not transitive anymore).

	NW(P,P_j)	**RN(P,P_j)**	**NE(P,P_j)**	**RW(P,P_j)**	**SP(P,P_j)**	**RE(P,P_j)**	**SW(P,P_j)**	**RS(P,P_j)**	**SE(P,P_j)**
NW(P_i,P)	NW	NW RN	N	NW RW	NW RN RW SP	N SL	W	W SH	U
RN(P_i,P)	NW RN	N	NE RN	NW RN RW SP	N SL	NE RN RE SP	W SH	U	E SH
NE(P_i,P)	N	NE RN	NE	N SL	NE RN RE SP	NE RE	U	E SH	E
RW(P_i,P)	NW RW	NW RN RW SP	N SL	W	W SH	U	SW RW	RW SP SW RS	S SL
SP(P_i,P)	NW RN RW SP	N SL	NE RN RE SP	W SH	U	E SH	SW RS RW SP	S SL	SE RS RE SP
RE(P_i,P)	N SL	NE RN RE SP	NE RE	U	E SH	E	S SL	SE RS RE SP	SE RE
SW(P_i,P)	W	W SH	U	SW RW	SW RS RW SP	S SL	SW	SW RS	S
RS(P_i,P)	W SH	U	E SH	RW SP SW RS	S SL	SE RS RE SP	SW RS	S	SE RS
SE(P_i,P)	U	E SH	E	S SL	SE RS RE SP	SE RE	S	SE RS	SE

Figure 17 Composition table for directions with neutral area

The same ideas can be applied for angular directions. In general, what is needed for the application of the algorithms is a] a set of direction relations for points and one for regions, b] rules for the inference of the relation between points given the relation between ancestor regions and finally c] composition rules.

7. CONCLUSIONS

The hierarchical representation of space has a strong psychological motivation (Hirtle and Jonides, 1985) and numerous computational advantages that have been exploited in a number of areas such as Data Structures (Guttman, 1984) and Wayfinding (Car and Frank, 1994). In this paper we focus on hierarchical spatial reasoning involving direction relations in 2D space. Although we have dealt with a set of projection-based direction relations often found in the literature, the methods of the paper are not relation-specific. They could be applied to alternative sets of relations with the appropriate rules of inference.

We present two complementary algorithms for spatial inference and inconsistency detection in hierarchically structured spatial databases: (1) the first achieves inference of direction relations between points through their ancestor regions, and, (2) the second performs inference through chains of common points and path consistency. For both algorithms we provide the corresponding inference rules and formulas for their cost. Because the algorithms generate the relations between all pairs of points they don’t need to be executed for each individual query, but only after the contents of the database are modified.

Hierarchical representations result in information loss with respect to flat representations. Some relations between points in different regions cannot be derived by inference. On the other hand, they facilitate query processing for queries involving objects within the same entity. Furthermore, in many cases, hierarchical representations are not just an option but a necessity. Even in a single system, data about the same or overlapping areas but from different sources are stored separately. This information may be incomplete or inconsistent, and inference mechanisms are required to explicate relations and remove inconsistencies.

As interoperability issues are solved, heterogeneous spatial databases and open GIS will soon become a reality. Such systems will store huge amounts of spatial data in various formats and of variable quality. Users will query the systems requiring fast and accurate results (and not answers of the form "A is north and south of B"). Spatial inference mechanisms will play an important role for the detection of inconsistencies in the data and the integration of the different systems.

REFERENCES

Allen, J.F. “Maintaining Knowledge about Temporal Intervals”. CACM 26 (11), pp. 832-843, 1983.

Car, A., Frank, A. “General Principles of Hierarchical Spatial Reasoning: The Case of Wayfinding”. In the Proceedings of the 6th International Symposium on Spatial Data Handling, 1994.

Chang, S.K., Shi, Q.Y. Yan C.W. “Iconic Indexing by 2-D String”. IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. PAMI-9, no 3, pp. 413-428, 1987.

Clementini, E., Sharma, J., Egenhofer, M. “Modeling Topological Spatial Relations: Strategies for Query Processing”. International Journal of Computer and Graphics, vol. 18 (6), pp. 815-822, 1994.

Dutta, S. “Qualitative Spatial Reasoning: A Semi-Qualitative Approach using Fuzzy Logic”. Proceedings of the First Symposium on the Design and Implementation of Large Spatial Databases 1989.

Egenhofer, M., Franzosa R. (1991) “Point Set Topological Relations”. International Journal of Geographic Information Systems, 5, pp. 161-174, 1991.

Egenhofer, M., Sharma, J. “Assessing the Consistency of Complete and Incomplete Topological Information”. Geographical Systems, Vol. 1, No 1, pp. 47-68, 1993.

GeoInformatica 1(3): 251-273.

Egenhofer, M., “Spatial SQL: A Query and Presentation Language”, IEEE Transactions on Knowledge and Data Engineering, Vol. 6(1), pp. 86-95, 1994.

Faloutsos, C., Kamel, I., “Beyond Uniformity and Independence: Analysis of R-trees Using the Concept of Fractal Dimension”, Proceedings of the 13th ACM PODS Symposium on Principles of Database Systems, 1994.

Frank, A.U., “Qualitative Spatial Reasoning about Distances and Directions in Geographic Space”, Journal of Visual Languages and Computing, Vol. 3, pp. 343-371, 1992.

Frank, A. U. “Qualitative Spatial Reasoning: Cardinal Directions as an Example”. To appear in the International Journal of Geographic Information Systems.

Freksa, C. “Using Orientation Information for Qualitative Spatial Reasoning”. Proceedings of the International Conference GIS - From Space to Territory: Theories and Methods of Spatio-Temporal Reasoning in Geographic Space, Pisa, Italy. Springer Verlag LNCS.

Glasgow, J.I. “Spatial Reasoning in Indeterminate Worlds”. Proceedings of the National Conference on Artificial Intelligence, 1994.

Glasgow, J.I., Papadias, D. “Computational Imagery”. Cognitive Science, 16, pp. 355-394, 1992.

Grigni, M., Papadias, D., Papadimitriou, C., “Topological Inference”, Proceedings of the International Joint Conference of Artificial Intelligence, 1995.

Guttman, A., “R-trees: a Dynamic Index Structure for Spatial Searching”, Proceedings of ACM SIGMOD International Conference on Management of Data, 1984.

Hernandez, D., “Qualitative Representation of Spatial Knowledge”, Springer Verlag LNAI, 1994.

Herskovits, A. “Language and Spatial Cognition”. Cambridge University Press, 1986.

Hirtle, S, Jonides, J. “Evidence of Hierarchies in Cognitive Maps”. Memory and Cognition 13, pp. 208-217, 1985.

Holmes, P.D., Jungert, E. “Symbolic and Geometric Connectivity Graph Methods for Route Planning In Digitized Maps”. IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. PAMI-14, no 5, pp. 549-565, 1993

Lee, S-Y., Yang, M-C., Chen, J-W. “Signature File as a Spatial Filter for Iconic Image Database”. Journal of Visual Languages and Computing, 3, pp. 373-397, 1992.

Mackworth, A, Freuder, E. “The Complexity of some Polynomial Network Consistency Algorithms for Constraint Satisfaction Problems”. Artificial Intelligence, 25, pp. 65-74, 1985.

Mukerjee, A., Joe, G., “A Qualitative Model for Space”. Proceedings of the National Conference on Artificial Intelligence, 1991.

Papadias, D., Sellis, T., “Qualitative Representation of Spatial Knowledge in Two-Dimensional Space”, Very Large Data Bases Journal, Special Issue on Spatial Databases, Vol 3(4), pp. 479-516, 1994.

Papadias, D., Sellis, T. “A Pictorial Query-By-Example Language”. Journal of Visual Languages and Computing, Special Issue on Visual Query Systems, Vol. 6, No. 1, 53 -72, 1995.

Papadias, D., Theodoridis, Y., Sellis, T., Egenhofer, M. “Topological Relations in the World of Minimum Bounding Rectangles: a Study with R-trees”. Proceedings of ACM SIGMOD International Conference on Management of Data, 1995.

GeoInformatica 1(3): 251-273.

Papadias, D., Theodoridis, Y. "Spatial Relations, Minimum Bounding Rectangles, and Spatial Data Structures", To appear in the International Journal of Geographic Information Systems.

Peuquet, D., Ci-Xiang, Z. "An Algorithm to Determine the Directional Relationship between Arbitrarily-Shaped Polygons in the Plane". Pattern Recognition, Vol 20, No 1, pp. 65-74, 1987

Roussopoulos, N., Faloutsos,C., Sellis, T. "An Efficient Pictorial Database System for PSQL". IEEE Transactions on Software Engineering, Special issue on Image Databases, pp. 639-650, 1988.

Sharma, J., Flewelling, D., Egenhofer, M.I., "A Qualitative Spatial Reasoner". In the Proceedings of the 6th International Symposium on Spatial Data Handling, 1994.

Sharma, J., "Heterogeneous Spatial Reasoning in Geographic Information Systems". Ph.D. Thesis, Department of Spatial Information Science and Engineering, University of Maine, 1996.

Sistla, P., Yu, C., Haddad, R. "Reasoning about Spatial Relationships in Picture Retrieval Systems". In the Proceedings of the 20th VLDB Conference, 1994.

Smith, T., Park, K., "Algebraic Approach to Spatial Reasoning". International Journal of Geographic Information Systems, Vol. 6, No. 3, pp. 177-192, 1992.

Topaloglou, T., "Qualitative Reasoning about Imprecise Spatial information". Proceedings of the 2nd ACM Workshop on GIS, 1994.

van Beek, P., Cohen, R. "Exact and Approximate Reasoning about Temporal Relations". Computational Intelligence 6, 132-144, 1990.

van Beek, P. "Reasoning About Qualitative Temporal Information". Artificial Intelligence, 58, pp. 297-326, 1992.