A Formal Definition of Binary Topological Relationships

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Abstract

The exploration of spatial relationships is a multi-disciplinary effort involving researchers from linguistics, cognitive science, psychology, geography, cartography, semiology, computer science, surveying engineering, and mathematics. Terms like *close* and *far* or *North* and *South* are not as clearly understood as the standard relationships between integer numbers. The treatment of relationships among spatial objects is an essential task in geographic data processing and CAD/CAM. Spatial query languages, for example, must offer terms for spatial relationships; spatial database management systems need algorithms to determine relationships. Hence, a formal definition of spatial relationships is necessary to clarify the users' diverse understanding of spatial relationships and to actually deduce relationships among spatial objects. Based upon such formalisms, spatial reasoning and inference will be possible.

The topological relationships are a specific subset of the large variety of spatial relationships. They are characterized by the property to be preserved under topological transformations, such as translation, rotation, and scaling. A model of topological relations is presented which is based upon fundamental concepts of algebraic topology in combination with set theory. Binary topological relationships may be defined in terms of the boundaries and interiors of the two objects to be compared. A formalism is developed which identifies 16 potential relationships. Prototypes are shown for the eight relationships that may exist between two objects of the same dimension embedded in the corresponding space.

1. Introduction

Queries in spatial databases, such as Geographic Information Systems, image data bases, or CAD/CAM systems, are often based upon the relationships among spatial objects. For example, in geographic applications typical spatial queries are "Retrieve all cities which are within 5 miles of the interstate highway I 95" or "Find all highways in the states adjacent to Maine." Current commercial query languages do not sufficiently support such queries, because these languages provide only tools to compare equality or order of simple data types, such as integers or strings. The incorporation of spatial relationships over spatial domains into the syntax of a spatial query language is an essential extension beyond the power of traditional query languages (Egenhofer and Frank, 1988).

Some spatial query languages support spatial queries with some spatial relationships; however, the diversity, semantics, completeness, and terminology of these relationships vary dramatically. While some terms may be specific to particular applications, in general all spatial relationships are founded upon fundamental principles of geometry. A consistent and least redundant approach requires that the common concepts are identified at the geometry level in the form of a fundamental set of spatial relationships. These generic relationships can then be applied for the definition of application-specific relationships. In CAD/CAM, for instance, the expressions *left* and *right* are preferred, while geographic applications use the terms *East* and *West* describing possibly the same relationships. The terms in both applications rely upon the same geometric concept: (1) the two objects of interest are either disjoint or neighbors, and (2) a strict order relation (‘$<$’ or ‘$>$’) defines the sequence of the objects along a one-dimensional carrier.

The development of a coherent, mathematical theory of spatial relations to overcome shortcomings in almost all geographic applications is one of the goals being investigated by the National Center for Geographic Information and Analysis (Abler, 1987). A formal definition is a prerequisite for the reasoning about the relationships among spatial objects. It is important to identify those crucial features which make humans distinguish one spatial situation from another, or which make them judge two situations as the same. A formal approach will be beneficial for several reasons: (1) The formalism serves as a tool to identify and derive relationships. Redundant and contradicting relations can be avoided such that a minimal set of fundamental relationships can be defined. (2) The formal methods can be applied to determine the relation between any two spatial objects. Algorithms to determine relationships can be specified exactly, and mathematically sound models will help to define formally the relationships. (3) The formalism is expected to help prove the completeness of the set of relationships. (4) The fundamental relations can be used to combine more complex relations.

Such a formal approach must be capable of dealing with spatial objects of various dimensions as well as objects in various spaces. Mapping two objects into a lower space should not affect their topological relationship, provided the object can exist in the lower space. Likewise, the projection into a higher space should not change the relationship between the two objects.

The scope of this paper is to provide a formal framework for reasoning about topological relationships, a particular subset of the wide range of spatial relationships. Topological notions include the concepts of continuity, closure, interior, and boundary, which are defined in terms of neighborhood relations. In this context, topological equivalence is considered a crucial criterion for comparing relationships among objects. Topological properties

often conflict with metric ones. It is important to keep in mind that topological equivalence does not preserve distances; therefore, distance is excluded from all considerations in this paper to avoid any confusion between metric and topology. Instead, the subsequent investigations will be based upon continuity which is described in terms of incidence and neighborhood.
The spatial objects considered within the context of this paper are restricted to the following subset:

- All objects are represented as simplicial complexes, a model for spatial objects which will be discussed in section 4..
- All objects are cells, i.e., their boundary is not empty.
- All objects are not self-intersecting.
- All objects are connected.
- All objects are of genus 0, i.e., they have no holes.

Moreover, the underlying space must be topological and open. Space, such as the closed surface of a globe, will not be considered.
A systematic approach is necessary to identify similar relationships and to discriminate dissimilar ones. The dimensions of the objects to be compared and the underlying space are crucial for the occurence of certain relationships. The higher the dimension of the space, the greater is the variety of relationships between two of its objects. Likewise, objects of a higher dimension have the potential for more relationships than objects of lower dimensions. For example, the set of relationships between two polygons has more distinct relationships than the set between two points. The cardinality of the set is then the Cartesian product of the dimensions of the two objects and the underlying space.
The relationships for which the formalism will be verified are characterized as follows:

- All relationships are invariant under topological transformations.
- The relationships are of Boolean type and can hold for exactly two objects (binary relations).

Only relationships between two n-dimensional objects in the corresponding n-dimensional space will be investigated. This is to show the similarities of relationships between two objects of the same dimension and considered a first step toward an object-oriented view of spatial relationships.
The rest of this paper is organized as follows: Section 2 discusses various formalisms to describe spatial relationships. In section 3 the notion of *topological relationships* is introduced and a motivation is given for the use of topological means to specify these relationships. Section 4 and 5 present a spatial data model necessary for the definition of topological relationships and an algebraic interpretation of fundamental spatial operations, respectively. A set of topological relationships is defined in terms of bounding and interior faces in section 6.

2. Formalisms for Spatial Relationships

Three classes of spatial relationships are discriminated which are based upon different spatial concepts (Pullar and Egenhofer, 1988). It appears natural for each class to develop an independent formalism describing the relationships.

- Topological relationships are invariant under topological transformations, such as translation, scaling, and rotation. Examples are terms like *neighbor* and *disjoint*.
- Spatial order and strict order relationships rely upon the definition of order and strict order, respectively. In general, each order relation has a converse relationship. For example, *behind* is a spatial order relation based upon the order of *preference* with the converse relationship *in_front_of*.
- Metric relationships exploit the existence of measurements, such as distances and directions. For instance, "within 5 miles from the interstate highway I 95" describes a corridor based upon a specific distance.

This classification is not complete since it does not consider *fuzzy* relationships, such as *close* and *next_to* (Robinson and Wong, 1987), or relationships which are expressions about the motion of one or several objects, such as *through* and *into* (Talmy, 1983). These types of relationships can be considered as combinations of several independent concepts. Motion, for example, may be seen as a combination of spatial and temporal aspects. So far, three different formal approaches for the definition of spatial relationships exist in the literature. The first one is based upon distance and direction in combination with the logical connectors AND, OR, and NOT (Peuquet, 1986). The relationship *disjoint (A, B)*, for example, is defined by the constraint that the distance from any point of object A to any point of B is greater than 0. This approach has two severe deficiencies: (1) It is not possible to model *inclusion* or *containment*, unless 'negative' distances are introduced. Peuquet defines the relationship *touching*, for example, by the distance which "equals to zero at a single location and is never less than zero" (Peuquet, 1986); however, by definition, distances are symmetric and a violation of this principle would lead to strange geometries. (2) The lack of appropriate computer numbering systems for geometric applications (Franklin, 1984) impedes the immediate application of coordinate geometry and distance-based formalisms for spatial relationships. The assumption that every space has a metric is unnecessarily complex and promotes the confusion about two different concepts—metric and topology. The formal definition of spatial relationships in the context of a geo-relational algebra is based upon the representation of spatial data in the form of point sets (Güting, 1988). Binary relationships are described by comparing the 'points' of two objects with conventional set operators, such as *equal* and *less than or equal*. For example, the relationship *inside (x, y)* is expressed by *points (x)* $\subseteq$ *points (y)*. This point set approach is in favour of raster representations in which each object can be represented as a set of pixels, but it is not easily applicable to vector representation. A serious deficiency inherent to the point set approach is that only a subset of topological relationships is covered with this formalism. While *equality*, *inclusion*, and *intersection* can be described, the point set model does not provide the necessary power to define *neighborhood* relationships. A crucial characteristic of neighborhood is that the *boundaries* of two objects have common parts, while the *interiors* do not. These distinct object parts cannot be distinguished with the point set model; therefore, pure point set theory is not applicable for the description of those relationships which rely upon interior or bounding parts only.

A third approach was developed for the representation of relationships among 1-dimensional intervals in a 1-dimensional space (Egenhofer, 1987b; Pullar and Egenhofer, 1988). It is based upon the intersection of the boundary and interior of the two objects to be compared and distinguishes only between "empty" and "non-empty" intersection. Table 1 shows the specifications of the minimal set of mutually excluding topological relationships among one-dimensional intervals.

(i1, i2)	$\partial \cap \partial$	$^\circ \cap ^\circ$	$\partial \cap ^\circ$	$^\circ \cap \partial$
disjoint	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
meet	$\neq \emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
overlap	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
inside	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
contains	$\emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
covers	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
coveredBy	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
equal	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\emptyset$

Table 1: The minimal set of topological relationships among intervals in a one-dimensional space described by the intersection of boundaries ($\partial \cap \partial$), interiors ($^\circ \cap ^\circ$), boundary with interior ($\partial \cap \; ^\circ$), and interior with boundary ($^\circ \cap \partial$).

This method is superior to the other two formalisms because it describes topological relations by purely topological properties. In this paper it will be shown that it can be generalized for objects of higher dimensions than only one-dimensional intervals.

3. Topological Relationships

Figure 1 shows an introductory example upon which the phenomena of topological relationships will be explained. The two objects A and B are such that humans would use terms like *overlap* or *intersect* in order to describe their relationship.

A particular characteristic by which the *overlap* relationship can be described is the relation among the object parts. For example, the boundaries

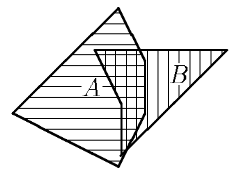

Figure 1: Two intersecting objects.

coincide in two points, both boundaries run through the opposite interior, and both interiors are partially identical. Figure 2 visualizes this concept comparing both boundaries, both interiors, boundary of one with interior of the other, and, reversely, interior of one with boundary of the other.

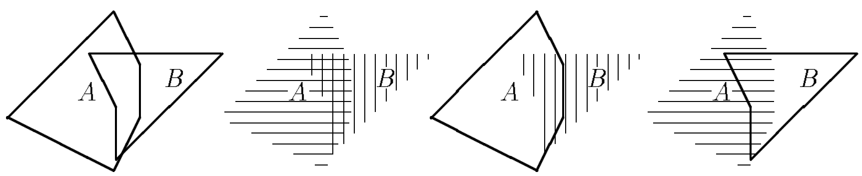

Figure 2: Comparing boundaries and interiors of two overlapping objects.

Another relationship between the same two objects is shown in Figure 3. Here, the common parts are only the coinciding boundary parts, while all other object parts do not have any commonality with the opposite parts.

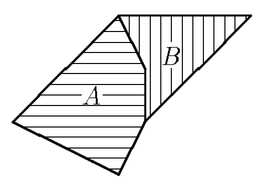

Figure 3: Two neighboring objects.

Compared to a similar situation shown in Figure 4, the only difference is that the common boundary has one edge less; however, this difference does not influence the judgement of the relationship between the two objects and humans will still use the same term describing the relationship.
Finally, Figure 5 shows two objects that are not neighbors. Similar to the previous modification, only a slight change was made; however, this time the two objects are not *neighbors* but *disjoint* from each other.

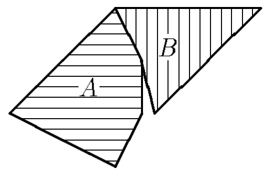

Figure 4: Two neighboring objects A and B sharing a common edge.

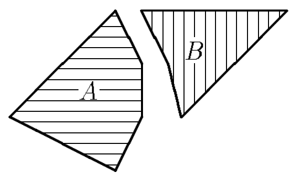

Figure 5: Two disjoint objects A and B.

These observations lead to two statements about the way to describe formally topological relationships:

Statement 1 *The topological relationship R between two spatial objects $o1$, $o2$ can be defined by comparing boundary and interior of $o1$ with the corresponding and opposite parts of $o2$.*

The second statement about the specification of spatial relationships can be derived from the pragmatic approach above:

Statement 2 *It is sufficient to consider "empty" and "non-empty" as values for the intersections of object parts.*

The two definitions guarantee complete coverage. Any further, more detailed relationship may be defined as a subset of one of them.
In order to define the crucial object parts *boundary* and *interior* for each object, a topological data model for spatial objects is needed.

4. A Model for the Representation of Spatial Data

In the mathematical theory of combinatorial topology, a sophisticated method has been developed to classify and formally describe point sets. This theory has been used for modeling spatial data (Corbett, 1979) and their composition. Recently, combinatorial topology was applied to spatial data models in Geographic Information Systems (GIS) (Frank and Kuhn, 1986; Herring, 1987), both for two-dimensional (Egenhofer, 1987a) and three-dimensional (Carlson, 1987) geometry. Their implementation demonstrated the simplicity of using a straight mathematical theory (Egenhofer *et al.*, 1989; Jackson, 1989). Subsequently, a brief introduction will be given of the concepts of the simplicial model relevant for the definitions of the topological relationships. More details, especially about operations upon simplicial complexes, are described elsewhere (Egenhofer *et al.*, 1989).

4.1 Simplex

Spatial objects are classified according to their spatial dimensions. For each dimension, a minimal object exists, called *simplex*. Examples for minimal spatial objects are 0-simplices representing nodes, 1-simplices which stand for edges, 2-simplices for triangles, 3-simplices for tetrahedrons, etc.
Any n-simplex is composed of (n+1) geometrically independent simplices of dimension (n-1). For example, a triangle, a 2-simplex, is bounded by three 1-simplices. These 1-simplices are geometrically independent if no two edges are parallel and no edge is of length 0 (Giblin, 1977).

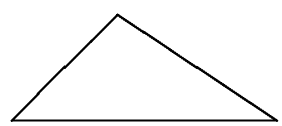

Figure 6: A 2-simplex composed of three 1-simplices.

A face of a simplex is any simplex that contributes to the composition of the simplex. For instance, a node of a bounding edge of a triangle is a face; another face of a triangle is any of its bounding edges.
A simplex S of dimension n has $\binom{n+1}{p+1}$ faces of dimension p ($0 \leq m \leq n$) (Schubert, 1968). For example, a 2-simplex has $\binom{2+1}{1+1}$ = 3 1-simplices as faces. Note that the n-simplex is a face of itself.

4.2 Simplicial Complex

A simplicial complex is a (finite) collection of simplices and their faces. If the intersection between two simplices of this collection is not empty, then the intersection is a simplex which is a face of both simplices. The dimension of a complex c is taken to be the largest dimension of the simplices of c.
The configurations in Figure 7, for example, are complexes, while Figure 8 shows three compositions which are not simplicial complexes.

4.3 Boundary

An important operation upon a n-simplex is *boundary*, denoted by ∂, which determines all (n-1)-faces of a simplex. The boundary of a n-complex is the (n-1)-chain of all (n-1)-simplices

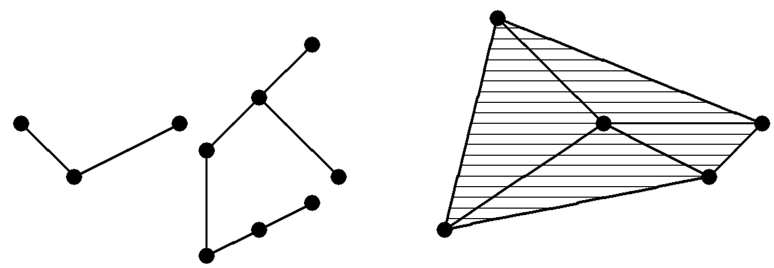

Figure 7: A 1- and a 2-complex.

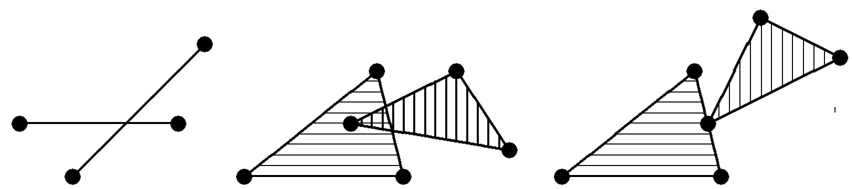

Figure 8: Three compositions which are not simplicial complexes.

The converse operation to boundary is *interior*, denoted by $^\circ$. Interior determines the set of all (n-1)-simplices which are not part of the boundary of a n-complex. Figure 9 shows a 2-complex with five bounding 1-faces and two interior 1-faces.
The property that two successive applications of boundary give the zero homomorphism is in agreement with the geometric notion that the boundary of a simplex is a closed surface.

4.4 Completeness Axioms

The simplicial model locates all spatial objects in the same world which is closed in analogy to the closed world assumption for non-spatial mini-worlds. The closed world assumption is extended by the two completeness principles for spatial data (Frank and Kuhn, 1986):

- Completeness of incidence: the intersection of two n-simplices is either empty or a face of both simplices.
- Completeness of inclusion: Every n-simplex is a face of a (n+1)-simplex. Hence, in a 2-dimensional space every node is either start- or end-point of an edge, and every edge is the boundary of a triangle.

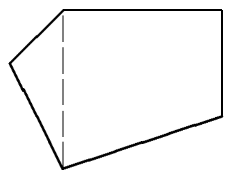

Figure 9: Boundary and interior of a 2-complex.

5. An Algebraic Approach

The algebraic interpretation of the boundary operation is particularly useful for the subsequent formal investigations. For this goal the orientation of a simplex is introduced, fixing the vertices to lie in a sequence. The orientation of a 0-simplex is unique; the two orientations of a 1-simplex can be interpreted as the direction *from* vertex A *to* vertex B and reverse *from* B *to* A (Figure 10); the orientations of a 2-simplex are *clockwise* or *counterclockwise*.

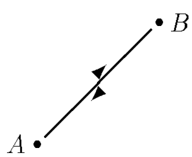

Figure 10: The two orientations of a 1-simplex.

Now suppose that the representation of the ordered n-simplex s_n be

$$s_n = \langle x_0, \cdots, x_n \rangle \tag{1}$$

then the boundary of s_n is determined by

$$\partial s_n = \sum_{i=0}^{n} (-1)^i \langle x_0, \cdots, \widehat{x}_i, \cdots, x_n \rangle \tag{2}$$

where $\widehat{x}_i$ denotes that the face x_i is to be omitted (Schubert, 1968). The bounding simplices form a chain which is an additive (i.e., free Abelian) group. Hence, the boundary of a simplicial complex c_n can be determined as the sum of the boundaries of all its simplices s_n.

$$\partial c_n = \sum \partial s_n \text{ if } s_n \in c_n \tag{3}$$

Figure 11 illustrates the following example: The two neighboring 2-simplices A_2 and B_2 have the following boundaries:

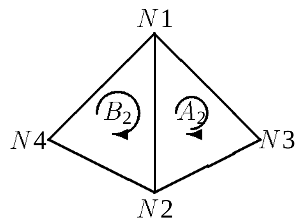

Figure 11: Calculating the boundary of the 2-complex C_2 from the boundaries of the two 2-simplices A_2 and B_2.

	simplex	∂
A_2	$\langle N1, N3, N2\rangle$	$\langle N3, N2\rangle - \langle N1, N2\rangle + \langle N1, N3\rangle$
B_2	$\langle N1, N2, N4\rangle$	$\langle N2, N4\rangle - \langle N1, N4\rangle + \langle N1, N2\rangle$

Table 2: Simplices and corresponding boundaries illustrated in Figure 11.

Then the complex C_2 formed by A_2 and B_2 has the boundary

$$\begin{aligned}
\partial C_2 &= \partial A_2 + \partial B_2 \\
&= \langle N3, N2\rangle - \langle N1, N2\rangle + \langle N1, N3\rangle + \langle N2, N4\rangle - \langle N1, N4\rangle + \langle N1, N2\rangle \\
&= \langle N3, N2\rangle + \langle N2, N4\rangle + \langle N4, N1\rangle + \langle N1, N3\rangle
\end{aligned}$$

5.1 Boundary Operator for Spatial Relationships

Unfortunately, the *boundary* operation as it is used in algebraic topology cannot be used immediately for the specification of spatial relationships. While the consideration of the faces of dimension $(n-1)$ is sufficient for the relationships among 1-complexes, it impedes the general treatment of relationships which are sometimes based upon common object parts of dimension $n-2$ or less. Figure 12 shows an example for a relationship which cannot be described by using *boundary* and *interior* in their purely mathematical sense. The intersection of the boundaries of the two-dimensional objects in one zero-dimensional object part is a crucial property of this neighborhood relationship; however, the intersection of the two boundaries does not identify any common parts, and applying the boundary operation upon these two objects does not help because boundary applied twice is always zero.

5.2 Bounding and Interior Faces

To overcome these shortcomings, the two operations *boundingFaces* and *interiorFaces* are introduced. They are modified *boundary* and *interior*

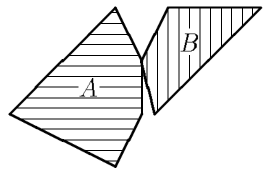

Figure 12: Two neighboring objects A and B sharing a common node.

operations which consider all faces down to dimension 0. Their algebraic definition is based upon the definition of boundary (equation 2) and skeleton (equation 4). The *r-skeleton* of a complex c_q, denoted by $c_q^{(r)}$, is defined as the union of all simplices of dimension at most r.

$$c_q^{(r)} = \bigcup_{i=0}^{r} s_i \in c_q \tag{4}$$

The *boundingFaces* of an n-dimensional complex c_n, denoted $\partial^f c_n$, is introduced as the (n-1) skeleton of the boundary of c_n.

$$\partial^f c_n = \bigcup_{r=0}^{n-1} c_n^{(r)} \in \partial c_n \tag{5}$$

The *interiorFaces* of an n-dimensional complex c_n, denoted $c_n^{\circ f}$, is the set of all faces of the n-skeleton of c_n which are not part of the *boundingFaces*.

$$c_n^{\circ f} = c_n^{(n)} \setminus \partial^f c_n \tag{6}$$

The dimension of the *boundingFaces* ∂^f of a complex c_n is defined to be the largest dimension of all faces in $\partial^f c_n$, i.e. n-1. In analogy, the dimension of the *interiorFaces* $c_n^{\circ f}$ is n. *BoundingFaces* and *interiorFaces* are sets upon which the traditional operations of set theory apply. In this context, only set intersection ($\cap$) will be needed.
Figure 13 shows the differences between *boundary* and *interior*, and *boundingFaces* and *interiorFaces*, respectively.

6. Formal Definition of Spatial Relationships

6.1 Formalism

Bounding and interior faces can be combined to form the four fundamental criteria of spatial relationships. These are: (1) common boundary parts as the intersection of *boundingFaces*, denoted by $\partial^f \cap \partial^f$, (2) common interior parts (${}^{\circ f} \cap {}^{\circ f}$), (3) boundary as part of the interior ($\partial^f \cap {}^{\circ f}$), and (4) interior

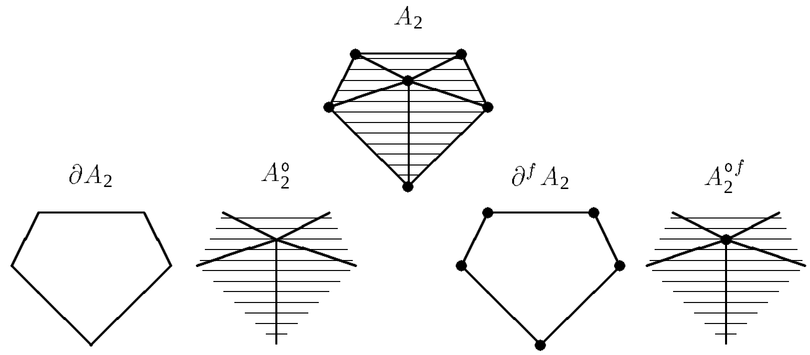

Figure 13: Boundary (∂), interior ($^\circ$), bounding faces (∂^f), and interior faces ($^{\circ f}$) of a complex A_2.

as part of the boundary ($^{\circ f} \cap \partial^f$). Subsequently, $\partial^f \cap \partial^f$ and $^{\circ f} \cap ^{\circ f}$ will be referred to as *corresponding* intersections, and $\partial^f \cap ^{\circ f}$ and $^{\circ f} \cap \partial^f$ as *opposite* intersections. With the binary values "empty" ($\emptyset$) and "non-empty" ($\neq \emptyset$) a total of 16 different specifications is given which provide the basis for the formal definition of the spatial relationships.

Lemma 1 *The 16 specifications as the Cartesian product of bounding and interior faces of two spatial objects with empty/non-empty values cover any possible constellation among them.*

Proof: Any two objects are completely described by bounding and interior faces. For the relationship between the two objects only the comparison with opposite object parts is significant, i.e. there are 2^2 comparisons. The Boolean values *empty* and *NOT empty* describe the full range of possible values for the intersections, i.e. each of the four constellations has two possible values resulting in 4^2 different specifications. □
The equivalence relation for two specifications is based upon the equivalence of the four components of a specification.

Lemma 2 *Two different pairs of objects $o1$, $o2$ and $o3$, $o4$ have the same relationship R if for each object pair the four intersections of the object parts have the same values, respectively.*

Proof: The equivalence relations *relationEqual* between two types of relationships is the conjunction of the equivalence relation *setEqual* for each of the 4 intersections of object parts. *SetEqual* be defined as follows:

```
OP setEqual: set_1 x set_2 -> boolean;
setEqual := (isEmpty (set_1) and isEmpty (set_2)) OR
            (NOT isEmpty (set_1) and NOT isEmpty (set_2));
```

SetEqual is an equivalence relation because it is reflexive, symmetric, and transitive.
Now *relationEqual* is defined for the four intersections of opposite object parts *o0_bound*, *o0_int*, *o1_bound*, *o1_int* in terms of *setEqual*:

```
OP relationEqual: o0_bound x o0_int x o1_bound x
                       o1_int -> boolean;
relationEqual := setEqual (o0_bound, o1_bound) AND
                 setEqual (o0_bound, o1_int) AND
                 setEqual (o0_int, o1_bound) AND
                 setEqual (o0_int, o1_int);
```

Since *setEqual* is an equivalence relation it is implied that *relationEqual* as the conjunction of the four intersections is an equivalence relation as well. □
A geometric interpretation of the abstract definition will be given below. It is not a matter of the definition of terms for the relationships—a systematic terminology $r_0 \dots r_{15}$ would provide the same service. Nevertheless, it is felt that meaningful names improve the understanding of the abstract definitions of the relationships.

6.2 n-Dimensional Relationships

This is to cover the relationships among volumes in 3-D, polygons in the 2-D plain, intervals along a line, and points in 0-D. Not all 16 potential relationships exist under this restriction. In a zero-dimensional space, for instance, the set of relationships between two 0-complexes is trivial since all 0-complexes are *equal*. Subsequently, a definition of the eight relationships *disjoint*, *meet*, *overlap*, *inside*, *contains*, *covers*, *coveredBy*, and *equal* is given in terms of *boundingFaces* and *interiorFaces*.

Definition 1 *If all four intersections among all object parts are empty, then the two objects are* disjoint.

Disjoint is linear, such that two objects are either disjoint or they are not. The specification for *not disjoint* follows immediately from the definition above, i.e., both objects must not share any common face. An interpretation of *disjoint* for 2- and 3-complexes in the corresponding spaces is given in Figure 14.

Definition 2 *If the intersection among the bounding faces is not empty, whereas all other 3 intersections are empty, then the two objects* meet.

The nature of *meet* is such that it only matters that the two objects share at least one common bounding face.
Figure 15 shows two examples of pairs of 2- and 3-complexes which *meet*. Several different types of *meet* relationships exist which can be distinguished according to the dimension p of the common bounding faces. The detailed *meet* relations are called *p-meet*. Recall that the dimension of the bounding faces is defined as the largest dimension of all faces. The dimension of the

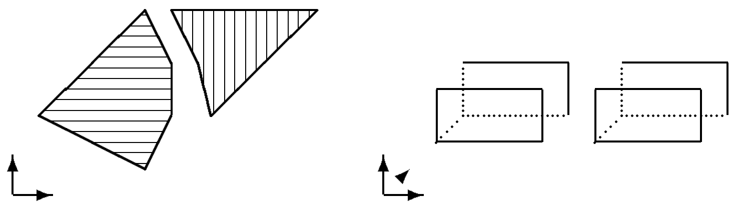

Figure 14: Two *disjoint* complexes in a 2- and 3-dimensional space, respectively.

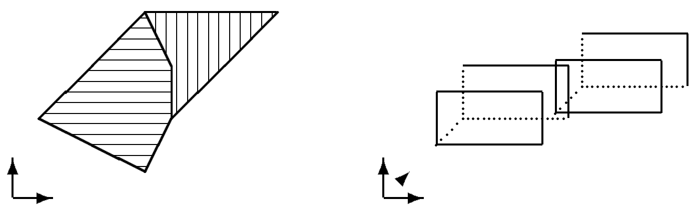

Figure 15: Two 2- and 3-complexes that *meet*.

intersection of two bounding faces is then the largest dimension of the faces being part of the intersection. Hence, there are n different types of boundary intersections between two n-dimensional objects.
Two n-dimensional objects can meet in n different ways. For example, the bounding faces of two 2-complexes can be of dimension 1 if the common part are one or several edges. Then the relationship is called 1-*meet*. The second *meet* relationship in 2-D, *0-meet*, requires that the dimension of the common bounding faces is 0 (i.e., the common bounding parts are only nodes). Figure 16 shows the difference between 1-*meet* and 2-*meet* for two pairs of 2-complexes.

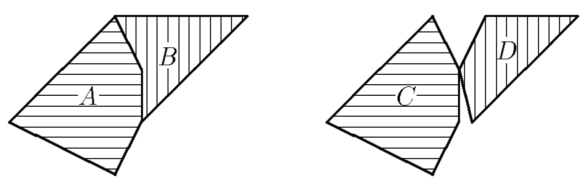

Figure 16: The 2 types of *meet* relationships among areal objects: 1-*meet* (A, B) and 0-*meet* (C, D).

Definition 3 *Two objects are* equal *if both intersections of bounding and interior faces are not empty while the two boundary-interior intersections* *are empty.*

Though this definition for equality may appear weak, it is sufficient for n-dimensional objects in an n-dimensional space.
For the sake of completeness, the stronger definition of equality is mentioned, too: Two objects are equal if they have the same bounding and interior faces. It is obvious that the former definition is a subset of this one. Due to the restriction that the objects and the underlying space have the same dimensions, any other constellations but *equal* are excluded under the requirement that the two opposite intersections are empty, while the corresponding intersections are not.

Definition 4 *An object A is* inside *of another object B if (1) A and B share interior, but not bounding faces, (2) if A has bounding faces which are interior faces of B, and (3) none of $B's$ bounding faces coincides with any of $A's$ interior faces.*

Inside has a converse relation *contains* which has the opposite definition of the boundary-interior intersections.

Definition 5 *An object A* contains *another object B if A and B share interior but no bounding faces; if B has bounding faces which are interior faces of A, and none of $A's$ bounding faces coincides with any of $B's$ interior faces.*

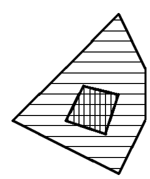

Figure 17: A 2-D object *inside* another 2-D object.

An integration relationship for *inside/contains* is *concur* which states that one of the two opposite intersections must be empty, while the other must not be.

Definition 6 *An object A* covers *another object B if both objects share common bounding and interior faces; if B has interior faces which are bounding faces of A; and if none of A's interior faces are part of B's boundary.*

Like *inside*, *covers* has a converse relationship, called *covered_by*, with corresponding specifications which are the same except for the reverse opposite intersections.

Definition 7 *An object A is* covered_by *another object B if both objects share common bounding and interior faces; if A has interior faces which are bounding faces of B; and if none of B's interior faces are part of A's boundary.*

In analogy to *meet*, several versions of *cover/covered_by* exist which distinguish by the dimension of the boundary intersection.

Definition 8 *Two objects* overlap *if they have common interior faces and the bounding faces have common parts with the opposite interior faces.*

This definition does not include any statement about the relation between the two boundaries. Indeed, *overlap* holds true no matter what the intersection of the two boundaries is. Figure 18 shows examples for overlapping lines and polygons with different boundary intersections.

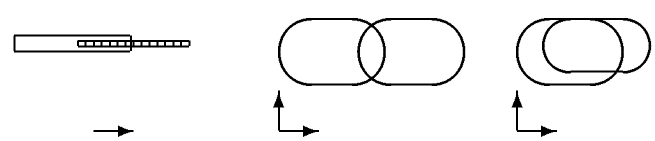

Figure 18: Overlapping complexes in 1- and 2-dimensional space.

7. Conclusions

A formalism for the definition of binary topological relationships between spatial objects was introduced. The formalism is based upon a sophisticated mathematical model for spatial data, the simplex theory. Crucial operations for the definitions of topological relationships are *boundingFaces* and *interiorFaces*, which are modifications of the traditional operators *boundary* and *interior*. The comparison of bounding and interior faces with the binary values *empty* and *non-empty* gave rise to 16 different specifications.
The specifications for the relationships between two n-dimensional objects in the corresponding n-dimensional space were investigated more thoroughly. A surprising regularity was obtained showing that the nature of topological relations is not erratic but rather systematic.
Further investigations are needed (1) to define the relationships of objects in higher-dimensional spaces, such as lines in 2-D or areas in 3-D, and (2) to verify that the specifications hold among objects of different dimensions as well. The Research Initiative 2 "Languages of Spatial Relations" of the recently established National Center for Geographic Information and Analysis will exploit the formalism proposed in this paper for more complex relationships.

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References

Abler, R., 1987. The National Science Foundation National Center for Geographic Information and Analysis. *International Journal of Geographical Information Systems*, 1(4):303–326.

Carlson, E., 1987. Three Dimensional Conceptual Modeling of Subsurface Structures. In: *ASPRS-ACSM Annual Convention*, pages 188–200, Baltimore, MD.

Corbett, J., 1979. *Topological Principles of Cartography*. Technical Report 48, Bureau of the Census, Department of Commerce.

Egenhofer, M., 1987a. Appropriate Conceptual Database Schema Designs For Two-Dimensional Spatial Structures. In: *ASPRS-ACSM Annual Convention*, pages 167–179, Baltimore, MD.

Egenhofer, M., 1987b. *Relations Between Intervals In A One-Dimensional Space*. Internal Documentation, University of Maine, Orono, Department of Surveying Engineering, Orono, ME.

Egenhofer, M. and Frank, A., 1988. Towards a Spatial Query Language: User Interface Considerations. In: D. DeWitt and F. Bancilhon, editors, *14th International Conference on Very Large Data Bases*, pages 124–133, Los Angeles, CA.

Egenhofer, M., Frank, A., and Jackson, J., 1989. A Topological Data Model for Spatial Databases. In: A. Buchmann, O. Günther, T. Smith, and Y. Wang, editors, *Symposium on the Design and Implementation of Large Spatial Databases, Lecture Notes in Computer Science, Vol. 409*, pages 271–286, Springer-Verlag, New York, NY.

Frank, A. and Kuhn, W., 1986. Cell Graph: A Provable Correct Method for the Storage of Geometry. In: D. Marble, editor, *Second International Symposium on Spatial Data Handling*, pages 411–436, Seattle, WA.

Franklin, W.R., 1984. Cartographic Errors Symptomatic of Underlying Algebra Problems. In: *International Symposium on Spatial Data Handling*, pages 190–208, Zurich, Switzerland.

Giblin, P., 1977 *Graphs, Surfaces, and Homology*. Halsted Press, John Wiley and Sons, New York, NY.

Güting, R., 1988. Geo-Relational Algebra: A Model and Query Language for Geometric Database Systems. In: J. Schmidt, S. Ceri, and M. Missikoff, editors, *Advances in Database Technology—EDBT '88, International Conference on Extending Database Technology, Venice, Italy*, pages 506–527, Springer-Verlag, New York, NY.

Herring, J., 1987. TIGRIS: Topologically Integrated Geographic Information Systems. In: N.R. Chrisman, editor, *AUTO-CARTO 8, Eighth International Symposium on Computer-Assisted Cartography*, pages 282–291, Baltimore, MD.

Jackson, J., 1989. Algorithms for Triangular Irregular Networks Based on Simplicial Complex Theory. In: *ASPRS-ACSM Annual Convention*, pages 131–136, Baltimore, MD.

Peuquet, D., 1986. The Use of Spatial Relationships to Aid Spatial Database Retrieval. In: D. Marble, editor, *Second International Symposium on Spatial Data Handling*, pages 459–471, Seattle, WA.

Pullar, D. and Egenhofer, M., 1988. Towards Formal Definitions of Topological Relations Among Spatial Objects. In: D. Marble, editor, *Third International Symposium on Spatial Data Handling*, pages 225–242, Sydney, Australia.

Robinson, V. and Wong, R., 1987. Acquiring Approximate Representations of Some Spatial Relations. In: N.R. Chrisman, editor, *AUTO-CARTO 8, Eighth International Symposium on Computer-Assisted Cartography*, pages 604–622, Baltimore, MD.

Schubert, H., 1968. *Topology*. Allyn and Bacon, Inc., Boston, MA.

Talmy, L., 1983. How Language Structures Space. In: H. Pick and L. Acredolo, editors, *Spatial Orientation: Theory, Research, and Application*, pages 225–282, Plenum Press, New York, NY.