Assessing the Consistency of Complete and Incomplete Topological Information[*]

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Abstract

High-level topological information about spatial objects can be described in terms of a set of binary topological relations between the objects, also called a scene description. The objects of interest are *spatial regions*, which are bounded objects that have a distinct identity and are homeomorphic to a 2-disk. The consistent integration of topological information relies inherently on the algebraic properties of the relations between the objects. Properties such as the converseness of pairs of relations and the composition of relations must be fulfilled for any combination of relations in order to guarantee that a scene description is free of internal topological contradictions so that it can be realized in $\mathbb{R}^2$. A rigorous computational method has been designed to reason about binary topological relations between spatial regions and to infer the consistency of complete and incomplete topological information. As a side-product, the method can be also used to refine incomplete observations. The method applies immediately in spatial query processing in geographic information systems to detect unsolvable queries prior to query execution and in data fusion to integrate independently collected information.

1 INTRODUCTION

The growing interest in scientific investigations of global climate change or the spread of AIDS across the world has dramatically increased the demand for systems that are capable of maintaining very large geographic data collections and making them available to scientists. State-of-the-art software systems for managing geographic data collections severely impede the timely evaluation and analysis of important data collections (French *et al.*, 1990, Smith and Frank, 1990), as most

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prominently exemplified by the delayed discovery of the "ozone hole." In order to analyze data with respect to the earth and its population, scientists need better access to many scientific data collections with large amounts of spatial data. The systems that manage the storage and retrieval of large geographic data sets must include *intelligent* mechanisms to deal with complex spatial concepts for data selection and data integration, i.e., computational methods that exploit spatial knowledge, rather than manipulate particular values of spatial data.

While many spatial inferences may appear trivial to humans—we handle them in our everyday life so frequently that we sometimes do not even recognize them as something special—they are extremely difficult to formalize in such a way that they could be implemented on a computer system (McCarthy, 1977, Bobrow *et al.*, 1986). Among the most serious deficiencies scientists face with today's database management systems is the lack of appropriate operators for manipulating the kinds of data encountered in scientific applications (French *et al.*, 1990). In geo-sciences, these operations are fundamentally spatial. The investigation of better suited spatial operators is germane to geographic analysis, because these operators are crucial when searching for data in a large spatial database, a task critical to the success of any scientific investigation, as demonstrated by the following two examples:

- Scientists who request access to specific subparts of remotely sensed images base their selections on *complex spatial criteria*, such as requests to find all scenes that cover islands in tropical seas with cliffs along the shore lines or scenes containing atolls. Such queries can be processed if additional data is integrated as metadata with the remotely sensed data, e.g., information about topography, political boundaries, and terrain, and if capabilities are available to reason about geographic space with some geographic common sense.

- Biologists' collections of herbarium specimens contain narrative descriptions of the sites where each specimen was found. Spatial analyses, e.g., about endangered species or the relations to soil types and climate, are severely hampered by the lack of automated methods to integrate the individual natural language descriptions of geographic spaces, e.g., by mapping the locations so that the *spatial relations* among them can be compared (McGranaghan and Wester, 1988, Futch *et al.*, 1992).

This paper addresses a particular problem within the realm of analyzing data collections based on complex spatial conditions. Conditions among spatial data are commonly expressed in terms of *spatial relations* (Frank and Mark, 1991) or *spatial prepositions* (Herskovits, 1986). Examples are *inside*, *north*, and *far* (Freeman, 1975, Peuquet, 1986). Such spatial relations are binary predicates, i.e., each relation holds between two objects, though, there are more complex ones, such as *between* which holds among three objects. The focus is here on *binary topological relations*. Topological relations are preserved under groups of transformations such as scaling, rotation, and translation and describe concepts of adjacency, containment, and intersection. Other kinds of spatial relations, not considered in this paper, are distance and direction relations (Peuquet and Ci-Xiang, 1987, Frank, 1992) and order relations (Kainz *et al.*, 1993).

The present investigations are part of a larger effort, the formalization and development of a comprehensive spatial reasoning system, similar to a human expert. Such formal systems will have multiple applications, e.g., to detect inconsistencies in spatial data collections, to design query optimizers for queries over multiple spatial conditions (Egenhofer and Sharma, 1992), as a starting point to investigate when two pairs of spatial objects have the same, or a similar, spatial relation, and as a base for extending common sense rule bases (Lenat *et al.*, 1990) with geographic knowledge. Related research efforts in spatial reasoning about geographic space include investigations about humans' cognitive maps (Gärling, 1989, Hirtle and Jonides, 1985) and how they acquire spatial knowledge (Kuipers, 1978). Complementary investigations focus on graphical applications such as reasoning about objects on raster images (Chang *et al.*, 1989, Lee and Hsu, 1992).

The scope of this paper is the evaluation of (symbolic) scene descriptions of spatial objects for topological consistency. A *scene description* consists of a set of spatial objects and their binary topological relations. The spatial objects may be geographic features, such as lakes, census districts, or market areas, that are bounded and have an identity. This model for geographic data is distinct from *fields* (Goodchild, 1992, Couclelis 1992). Topological consistency of a scene description means that there is no internal contradiction among the individual relations due to their properties. Topological consistency in geographic information systems (GISs) is usually dealt with at the conceptual level of nodes, lines, and areas (Corbett, 1979, White, 1984, Herring, 1987, Egenhofer *et al.*, 1989). It ensures that their interrelationships are complete, for instance, that an area is bounded by a set of closed sequences of non-intersecting lines. These issues are critical for consistent implementations of geometry in GISs (Frank and Kuhn, 1986, Herring, 1991). This paper considers topological consistency at a higher level, independent of the way spatial objects are encoded. The *visual inspection* of the topological consistency of a scene description is seemingly straightforward—one tries to draw a corresponding configuration and, if successful, this showed that the topological description was consistent. Obviously, such an interpretation is influenced by the observer's subjectiveness and many different graphical representations may be depicted for the same topological relation so that it becomes increasingly difficult to analyze and compare scene descriptions with many objects. Since visual analysis is informal, it does not lead to immediate implementations in a computer. This paper uses an alternative approach to visual inspection. It is based on a mathematical model for binary topological relations so that the evaluation of the consistency of a scene description becomes a computational process; therefore, this model can be implemented as a computer program. We build on previous work that focused on a mathematical formalism for point-set topological relations between spatial regions (Egenhofer and Franzosa, 1991), a definition of the composition of binary topological relations (Egenhofer, 1991b), and initial studies on the use of the composition to reason about topological relations (Egenhofer and Sharma, 1992, Smith and Park, 1992).

The computational evaluation of the topological consistency of a scene description may be straightforward if all topological relations are related by some standard properties such as transitivity. For example, the description "A *contains* B, and B *contains* C, and C *contains* D, and A *meets* D" is obviously inconsistent because, by the transitive property of contains, A must *contain* D, which contradicts with the statement that A *meets* D. More difficult are such evaluations if a larger set of objects is involved or not only transitive relations occur. This paper makes use of such properties of the binary topological relations as symmetry, transitivity, converseness, and composition to formalize the inference of new spatial information in a general setting. The following example demonstrates how the composition and other properties of topological relations can be used to infer unknown topological information. Given a scene description with three 2-dimensional objects, A, B, and C, and the binary topological relations *A contains B* and *B contains C*. The first inference is that all objects must be *equal* to themselves. Second, transitivity of the containment relation implies that *A contains C* as well. Finally, the fact that *A contains B* implies the converse relation that B is *inside* of A—with corresponding inferences for the other relations. Thus the two topological relations *A contains B* and *B contains C* are sufficient to describe uniquely this configuration of three objects and no other relations are required. Such conclusions depend on the particular relations—for example, there is no unique geometric interpretation for the description "A *contains* B and B *inside* A"—and other combinations of relations may require different inference modes than just transitivity and converseness. For any scene description, the set of n^2 binary topological relations between the n objects is redundant since some of these topological relations are always implied by others. The result has multiple applications in geographic analysis and spatial data handling such as (1) detecting inconsistencies in spatial data collections, (2) optimizing spatial queries over topological relations, and (3) serving as a means to determine if two scene descriptions have the same or similar spatial relations.

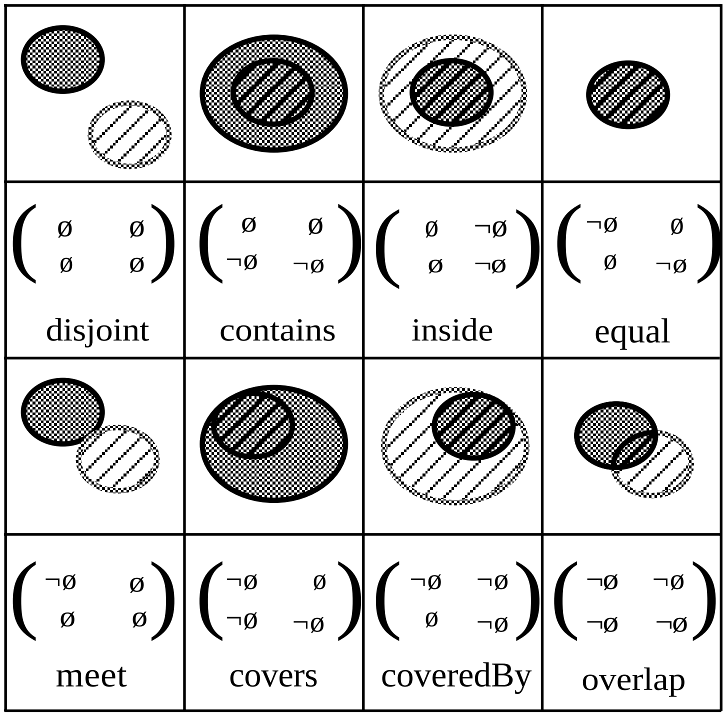

Figure 1: The 4-intersections for the eight topological relations between two spatial regions without holes and their geometric interpretations.

The remainder of the paper is structured as follows: Section 2 briefly describes the 4-intersection, a mathematical model for binary topological relations between spatial regions, and analyzes the properties of these relations and their composition. Section 3 presents a model for the representation of the topology of a scene, for which topological consistency constraints are formalized (Section 4). Subsequently, an algorithm is developed to evaluate the topological consistency of a completely observed scene description (Section 5). These concepts are generalized in Section 6, where scene descriptions with incomplete topological information are analyzed. Section 7 summarizes the results and discusses ongoing efforts that build on them.

2 BINARY TOPOLOGICAL RELATIONS BETWEEN SPATIAL REGIONS BASED ON THE 4-INTERSECTION

The usual concepts of point set topology with open and closed sets are assumed (Alexandroff, 1961, Spanier, 1966). The interior of a set A, denoted by A°, is the union of all open sets in A.

The closure of A, denoted by $\overline{A}$, is the intersection of all closed sets of A. The exterior of A with respect to the embedding space $\mathbb{R}^2$, denoted by A^-, is the set of all points of $\mathbb{R}^2$ not contained in A. The boundary of A, denoted by ∂A, is the intersection of the closure of A and the closure of the exterior of A. The objects of concern in this paper are *spatial regions*, which are defined as homogeneously 2-dimensional point sets with connected boundaries. The definition of binary topological relations between two spatial regions, A and B, is based on the four intersections of A's boundary (∂A) and interior (A°) with the boundary (∂B) and interior (B°) of B (Egenhofer and Franzosa, 1991). By considering the values empty ($\emptyset$) and non-empty ($\neg\emptyset$) for the 4-intersection, one can distinguish sixteen binary topological relations. Eight topological relations can be realized for two spatial regions if they are embedded in 2-D (Egenhofer and Franzosa, 1991). Note that the metric of the spatial regions, such as their shape—whether convex or concave—does not matter as these definitions are based on purely topological principles (boundary and interior), which are independent of other geometric properties. We call the eight topological relations between two spatial regions *disjoint*, *meet*, *equal*, *inside*, *contains*, *covers*, *coveredBy*, and *overlap* (Figure 1), though any other notation such as $R_0 \ldots R_7$ would do the same service. These terms should not be given any semantic, cognitive, or linguistic interpretation as they refer only to the formal, mathematical definitions. Linkages between formal mathematics of this model and human cognition about space are investigated elsewhere (Mark and Egenhofer, 1992). Readers should refer to the formal definitions with respect to their boundary and interior intersections, rather than trying to associate intuitive interpretations with these terms. The set of eight binary topological relations provides a complete coverage, i.e., any possible configuration is treated by one 4-intersection, and it is mutually exclusive, i.e., exactly one of these topological relations holds true between any two spatial regions (Egenhofer and Franzosa, 1991). Subsequently, $\mathcal{R}$ will refer to the set of binary topological relation between two spatial regions and r_i and r_j will be distinct elements of $\mathcal{R}$.

A 2 $\times$ 2 matrix, $\mathcal{M}$, called the 4-intersection, concisely represents the criteria (Equation 1).

$$\mathcal{M}\ (A, B) = \begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^\circ \\ A^\circ \cap \partial B & A^\circ \cap B^\circ \end{pmatrix} \tag{1}$$

To refer to a particular intersection between two spatial regions, the short form $\mathcal{M}[\_\_]$ will be used, e.g., $\mathcal{M}[\partial^\circ]$ to refer to the value of the boundary-interior intersection. The topological relation between two spatial regions A and B will be denoted by $r\ (A, B)$.

Some basic matrix operations apply to the 4-intersection, such as the transposition $\mathcal{M}^T$, which can be used to determine properties of the topological relations by analyzing the 4-intersections. $\mathcal{M}^T(A, B)$ describes the 4-intersection of $r\ (B, A)$, the topological relation converse to $r\ (A, B)$.

- A binary relation r_i is called *symmetric* if $r_i\ (A, B)$ implies $r_i\ (B, A)$. Based on the 4-intersection, a binary topological relation r_i is symmetric if

$$\mathcal{M}_i[\partial^\circ] \quad = \quad \mathcal{M}_i[^\circ\partial]. \tag{2}$$

or, more generally, $\mathcal{M}_i \ = \ \mathcal{M}_i^T$.

Example 1 *Meet* is symmetric, because

$$\mathcal{M}_{meet}[\partial^\circ] \quad = \quad \mathcal{M}_{meet}[^\circ\partial] \quad = \quad \emptyset$$

□

- A pair of binary relations, r_i and r_j, is called *converse* if $r_i\ (A,B)$ implies $r_j\ (B,A)$. In terms of the 4-intersection, the converse property of two binary topological relations can be described formally by

$$\begin{aligned}
\mathcal{M}_i[\partial\partial] &= \mathcal{M}_j[\partial\partial] \quad \wedge \\
\mathcal{M}_i[\partial^\circ] &= \mathcal{M}_j[^\circ\partial] \quad \wedge \\
\mathcal{M}_i[^\circ\partial] &= \mathcal{M}_j[\partial^\circ] \quad \wedge \\
\mathcal{M}_i[^{\circ\circ}] &= \mathcal{M}_j[^{\circ\circ}]
\end{aligned} \tag{3}$$

  or, concisely, $\mathcal{M}_i = \mathcal{M}_j^T$.

Example 2 *Inside* and *contains* are converse, because

$$\begin{aligned}
\mathcal{M}_{inside}[\partial\partial] &= \mathcal{M}_{contains}[\partial\partial] &= \emptyset \quad \wedge \\
\mathcal{M}_{inside}[\partial^\circ] &= \mathcal{M}_{contains}[^\circ\partial] &= \neg\emptyset \quad \wedge \\
\mathcal{M}_{inside}[^\circ\partial] &= \mathcal{M}_{contains}[\partial^\circ] &= \emptyset \quad \wedge \\
\mathcal{M}_{inside}[^{\circ\circ}] &= \mathcal{M}_{contains}[^{\circ\circ}] &= \neg\emptyset
\end{aligned}$$

□

Subsequently, we will introduce some properties of the topological relations. They are elements of a *relation algebra* (Maddux, 1990) over topological relations and are prerequisites for a formal analysis of the topological consistency of a scene description.

2.1 Universal Topological Relation

The universal topological relation $\mathcal{U}$ is the union of all possible topological relations and holds true for the topological relation between any two spatial regions.

$$\begin{aligned}
\mathcal{U} &= \text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{inside} \vee \text{coveredBy} \vee \text{contains} \vee \text{covers} \vee \text{overlap} \\
&= \bigcup r_i
\end{aligned}$$

2.2 Empty Topological Relation

The empty topological relation, $\emptyset$, is introduced to denote that a topological relation cannot be realized, because it contradicts with respect to the other topological relations within the same configuration.

2.3 Converseness

From the 4-intersections (Figure 1) and Equation 3 it can be derived that *each* of the eight possible topological relations $r\ (A,B)$ between two spatial regions has a *converse* topological relation $\breve{r}\ (B,A)$. These pairs of converse topological relations are:

- disjoint (A, B) = disjoint (B, A)
- meet (A, B) = meet (B, A)
- equal (A, B) = equal (B, A)
- overlap (A, B) = overlap (B, A)

- inside (A, B) = contains (B, A)
- contains (A, B) = inside (B, A)
- covers (A, B) = coveredBy (B, A)
- coveredBy (A, B) = covers (B, A)

2.4 Composition

The *composition* of two topological relations, r_i (A, B) and r_j (B, C), over a common spatial region B, denoted by $r_i; r_j$, allows for the derivation of the topological relation r_k between A and C. The composition table (Table 1), which was formally derived based on the transitivity of empty and non-empty intersections (Egenhofer, 1991b), shows the outcome of all 64 possible compositions among all 8 topological relations. In addition, any composition with the empty topological relation is defined to be the empty topological relation.

	disjoint d (B, C)	meet m (B, C)	equal e (B, C)	inside i (B, C)	coveredBy cB (B, C)	contains ct (B, C)	covers cv (B, C)	overlap o (B, C)
disjoint d (A, B)	$\mathcal{U}$	d, m, i, cB, o	d	d, m, i, cB, o	d, m, i, cB, o	d	d	d, m, i, cB, o
meet m (A, B)	d, m, ct, cv, o	d, m, e, cB, cv, o	m	i, cB, o	m, i, cB, o	d	d, m	d, m, i, cB, o
equal e (A, B)	d	m	e	i	cB	ct	cv	o
inside i (A, B)	d	d	i	i	i	$\mathcal{U}$	d, m, i, cB, o	d, m, i, cB, o
coveredBy cB (A, B)	d	d, m	cB	i	i, cB	d, m, ct, cv, o	d, m, e, cB, cv, o	d, m, i, cB, o
contains ct (A, B)	d, m, ct, cv, o	ct, cv, o	ct	e, i, cB, ct, cv, o	ct, cv, o	ct	ct	ct, cv, o
covers cv (A, B)	d, m, ct, cv, o	m, ct, cv, o	cv	i, cB, o	e, cB, cv, o	ct	ct, cv	ct, cv, o
overlap o (A, B)	d, m, ct, cv, o	d, m, ct, cv, o	o	i, cB, o	i, cB, o	d, m, ct, cv, o	d, m, ct, cv, o	$\mathcal{U}$

Table 1: The 64 compositions of the binary topological relations r (A, B) and r (B, C) with e = equal, m = meet, d = disjoint, i = inside, ct = contains, cv = covers, and cB = coveredBy (Egenhofer, 1991b).

The composition table verifies immediately a number of properties that are part of a relation algebra (Tarski, 1941). Any composition with *equal* results in the original topological relation; therefore, *equal* is the *identity relation*, denoted by $\mathcal{I}$. Only compositions with the converse relations may result in $\mathcal{I}$. On the other hand, any composition with the universal topological relation results in the universal topological relation. This can be easily derived from the composition table since for each topological relation r_i, the union of all compositions $r_i; r_j$, with $r_j = \mathcal{U}$, contains all possible topological relations. $r_i;\mathcal{U} = \mathcal{U}$ implies that $\mathcal{U};\mathcal{U} = \mathcal{U}$ as well. The composition of two relations is equal to the converse of the composition of the converse relations, applied in reverse order, i.e., $r_i; r_j = \breve{r}_j; \breve{r}_i$. Finally, the composition distributes over disjunctions (Maddux, 1990), i.e.,

$$r_i \; ; \; (r_j \vee r_k) \quad = \quad (r_i \; ; \; r_j) \vee (r_i \; ; \; r_k) \tag{4}$$

Example 3

$$\begin{aligned} \text{meet; (coveredBy} \vee \text{covers)} &= \text{(meet; coveredBy)} \vee \text{(meet; covers)} \\ &= \text{(meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap)} \vee \text{(disjoint} \vee \text{meet)} \\ &= \text{disjoint} \vee \text{meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap} \end{aligned}$$

□

The analysis of the composition table reveals also a number of particular properties of certain compositions:

- The composition is *unique* if it results in a single (non-empty) relation.

  Example 4 *meet* ; *contains* = *disjoint* □

- If $r_k = r_i; r_j$ is unique and $r_i = r_j = r_k$, then the relation is *transitive*.

  Example 5 *inside* ; *inside* = *inside* □

- If the composition results in the universal topological relation, then it is said to be *undetermined*.

  Example 6 *overlap* ; *overlap* = $\mathcal{U}$ □

- The composition is *underdetermined* if it is not undetermined and results in more than one possible relation.

  Example 7 *contains* ; *meet* = *contains* ∨ *covers* ∨ *overlap* □

3 SCENE REPRESENTATIONS

A generally applicable method to evaluate formally the topological consistency of a scene description must be based on a formal representation of a scene. This section introduces such a model.

The topology of a scene with n spatial regions is described by n^2 binary topological relations. The set of topological relations can be abstracted to a *directed graph*, in which each *node* represents an object and each *directed edge* between two nodes stands for a binary topological relation. Any two nodes within a graph are linked by a *directed path*, i.e., a sequence of edges that point in the same direction. The number of edges along a path is called the *path length*. When applied to topological relations between spatial regions, each node in the graph corresponds to a distinct spatial region and each directed edge with the corresponding label t_{IJ} stands for the binary topological relation $r\ (I, J)$ between the two spatial regions I and J. Within such a graph of n spatial regions, all n^2 binary topological relations of a scene description form a *complete directed graph*, i.e., there is a directed edge connecting any ordered pair of nodes. Figures 2 and 3 respectively show a graphical representation of a scene description and the corresponding graph. The labels of the nodes refer to

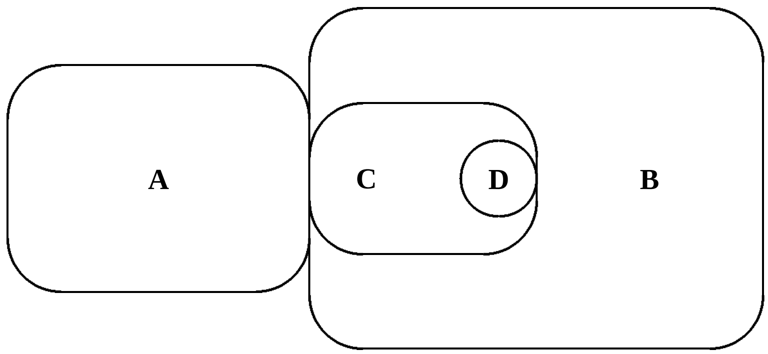

Figure 2: Example of a consistent scene with four spatial regions.

the (unique) identifiers of the spatial regions, while the labels of the edges refer to their topological relations.

This model of the topology of a scene has several advantages over a graphical representation. First, it is a symbolic representation of the topology of a scene and, therefore, can be formally analyzed without the need for a (subjective) visual or graphical inspection. Second, it allows us to consider only topological information without additional spatial relations such as directions, ratios of sizes, and relative distances, which are typically implied by graphical representations. Third, one can also model *incomplete* and *uncertain* information, which is an extremely difficult task with graphical representations. The advantages of the network representation are contrasted by the possibility of *inconsistent* scene descriptions, i.e., scene descriptions that cannot be realized in a particular space, because the intrinsic geometric constraints among multiple spatial regions have been absorbed in the network. For example, it is possible to construct a network for the description of a scene that "A is inside of B and B is inside of A" by labeling the directed edges between two nodes with the corresponding topological relations, though no geometric interpretation can be made for this scene description. Unlike a graphical sketch, the graph of topological relations has no implicit mechanisms to enforce topological consistency; therefore, explicit knowledge is necessary to evaluate the consistency of a scene description and to provide the same level of usefulness as a graphical representation does. Given a means to evaluate the consistency of any arbitrary network of topological relations, this model is more powerful than a graphical representation, because it is formal so that it can be implemented on a computer.

4 TOPOLOGICAL CONSISTENCY CONSTRAINTS

This section introduces a comprehensive method for the integration and comparison of individual topological relations. The basis for the integration is that there must be no logical contradiction among topological relations describing the same scene. Logical consistency is based on the algebraic properties of the individual relations, primarily their composition.

Given the scene representation as a directed graph, topological consistency constraints can be formulated as *constraint satisfaction problems* (Maddux, 1990) over a network of binary topological relations. To provide a consistent network of binary relations, the nodes, edges, and sequences of edges must fulfill three constraints: the graph must be node-consistent, arc-consistent, and path-consistent (Mackworth, 1977). If all three levels of consistency are fulfilled for the topological relations, a scene description is topologically consistent.

The trivial constraints for binary topological relations are immediately implied by the properties of the topological relations (Figure 3): (1) Each node N must have a self loop, denoting the identity relation $\mathcal{I}$, to ensure *node consistency*. (2) For each directed edge from N to M, representing the topological relation r (N, M), there must be an edge in the reverse direction, from M to N, with the implied converse topological relation $\breve{r}$ (M, N). If all edges in the network fulfill this constraint and if the network is node-consistent, the network is called *arc-consistent*.

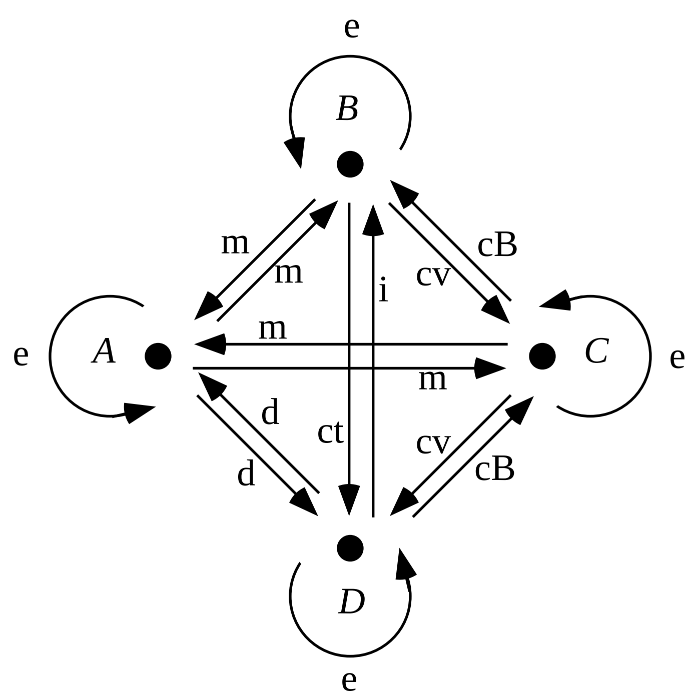

Figure 3: The constraint network for the configuration in Figure 2.

In terms of the constraint network, the directed edges must be labeled as follows:

$$\forall_i \ t_{ii} = \mathcal{I} \quad (5)$$

$$\forall_{i,j} \ t_{ij} = \breve{t}_{ji} \quad (6)$$

While these two constraints provide a certain level of consistency, they are still insufficient to guarantee topological consistency for any scene description.

Example 8 Figure 4 shows a network that fulfills the criteria for node- and arc-consistency, but is topologically inconsistent, because the composition of *disjoint ; contains* = *overlap* contradicts the composition table. □

The constraint that guarantees the consistency of the compositions has been called *path consistency* (Mackworth, 1977). It is based on the fact that within a topological constraint network, several paths can be found to get from one node to another. For example, in figure 3, nodes A and B can be reached over various paths: (1) directly, (2) via C, (3) via D, (4) via C *and* D, or (5) via D *and*

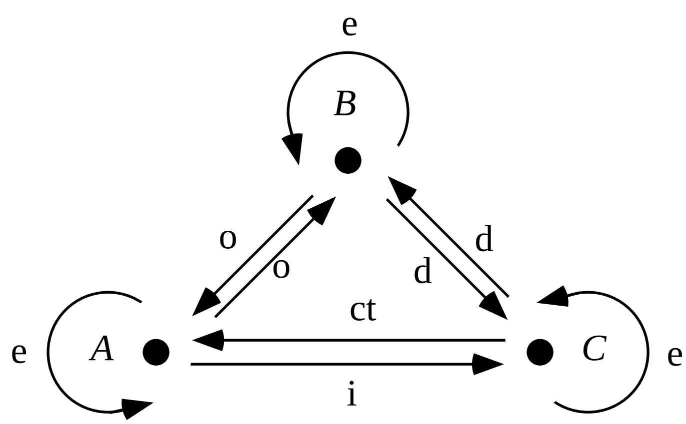

Figure 4: A constraint network that is node- and arc-consistent, but not path-consistent.

C. While there is a multitude of possible paths connecting any two nodes A and B, it is sufficient to consider all compositions of path length 2 that connect A to B to infer the consistency of their relation (Montanari, 1974). In order to be a path-consistent scene description, the corresponding network must be arc-consistent and each topological relation must coincide with its induced relation. In terms of the constraint network of $A \ldots N$ relations, the induced topological relation is equal to the intersection of all possible compositions of path length 2, i.e., the additional constraint for path consistency is:

$$\forall_{i,j} \ \ t_{ij} \ = \ t_{iA};t_{Aj} \ \cap \ t_{iB};t_{Bj} \ \cap \ \ldots \ \cap \ t_{iN};t_{Nj} \tag{7}$$

(where composition binds closer than intersection), or briefly

$$\forall_{i,j} \ \ t_{ij} \ = \ \bigcap_{k=A}^{N} t_{ik};t_{kj} \tag{8}$$

Example 9 Figure 5 shows a node- and arc-consistent network of a scene description with three objects, A, B, and C.

The nine induced relations are calculated as follows:

t_{AA} = equal; equal ∩ meet; meet ∩ meet; meet
= equal ∩ (disjoint ∨ meet ∨ equal ∨ coveredBy ∨ covers ∨ overlap) ∩
disjoint ∨ meet ∨ equal ∨ coveredBy ∨ covers ∨ overlap
= equal

t_{AB} = equal; meet ∩ meet; equal ∩ meet; coveredBy
= meet ∩ meet ∩ (meet ∨ inside ∨ coveredBy ∨ overlap)
= meet

t_{AC} = equal; meet ∩ meet; covers ∩ meet; equal
= meet ∩ (disjoint ∨ meet) ∩ meet
= meet

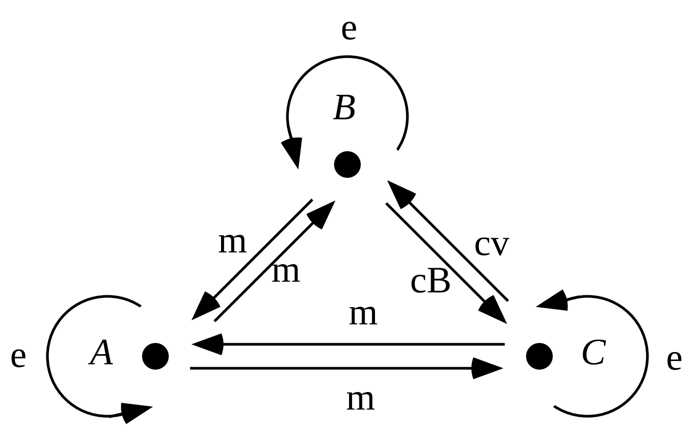

Figure 5: A path-consistent constraint network.

$$
\begin{aligned}
t_{BA} &= \text{meet; equal} \;\cap\; \text{equal; meet} \;\cap\; \text{covers; meet} \\
&= \text{meet} \;\cap\; \text{meet} \;\cap\; (\text{meet} \vee \text{contains} \vee \text{covers} \vee \text{overlap}) \\
&= \text{meet} \\
t_{BB} &= \text{meet; meet} \;\cap\; \text{equal; equal} \;\cap\; \text{covers; coveredBy} \\
&= (\text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \;\cap\; \text{equal} \;\cap \\
&\quad\; (\text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \\
&= \text{equal} \\
t_{BC} &= \text{meet; meet} \;\cap\; \text{equal; covers} \;\cap\; \text{covers; equal} \\
&= (\text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \;\cap\; \text{covers} \;\cap\; \text{covers} \\
&= \text{covers} \\
t_{CA} &= \text{meet; equal} \;\cap\; \text{coveredBy; meet} \;\cap\; \text{equal; meet} \\
&= \text{meet} \;\cap\; (\text{disjoint} \vee \text{meet}) \;\cap\; \text{meet} \\
&= \text{meet} \\
t_{CB} &= \text{meet; meet} \;\cap\; \text{coveredBy; equal} \;\cap\; \text{equal; coveredBy} \\
&= (\text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \;\cap\; \text{coveredBy} \;\cap\; \text{coveredBy} \\
&= \text{coveredBy} \\
t_{CC} &= \text{meet; meet} \;\cap\; \text{coveredBy; covers} \;\cap\; \text{equal; equal} \\
&= (\text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \;\cap \\
&\quad\; (\text{disjoint} \vee \text{meet} \vee \text{equal} \vee \text{coveredBy} \vee \text{covers} \vee \text{overlap}) \;\cap\; \text{equal} \\
&= \text{equal}
\end{aligned}
$$

□

Based on the notion of a constraint network, a topologically consistent scene description is evaluated in the following steps:

- Construct a node-consistent network from the initial network of topological relations:

$$\forall_i \; t'_{ii} \quad := \quad t_{ii} \cap \mathit{equal} \tag{9a}$$

$$\forall_{ij \mid i \neq j} \; t'_{ij} \;\; := \;\; t_{ij} \tag{9b}$$

- Construct an arc-consistent network from a node-consistent network of topological relations:

$$\forall_{ij} \; t''_{ij} \;\; := \;\; t'_{ij} \cap \breve{t}'_{ji} \tag{10}$$

- Construct a path-consistent network from an arc-consistent network of topological relations:

$$\forall_{ij} \; t'''_{ij} \;\; := \;\; \bigcap_{k=A}^{N} (t''_{ik} \; ; \; t''_{kj}) \tag{11}$$

5 COMPLETELY OBSERVED SCENE DESCRIPTIONS

This section assumes that a scene description is completely observed. In order to define a completely observed scene description, it is necessary to introduce the cardinality of a relation.

Definition 1 The *cardinality* of a relation, denoted by $\#t_{ij}$, is the (non-negative) number of relation values in t_{ij}.

Some important properties of the cardinality of topological relations are given below:

- A topological relation has cardinality 0 if and only if it is the empty topological relation $\emptyset$.
- Every non-empty relation has a positive cardinality.
- The cardinality of a single relation value is 1.
- The cardinality of the union of two topological relations, $\#(t_{ij} \vee t_{kl})$, is the sum of the cardinalities of each relation minus the number of relation values common to t_{ij} and t_{kl}.
- The range of the cardinality of a topological relation is between 0 ($\#\emptyset$) and 8 ($\#\mathcal{U}$).

Definition 2 A relation is *completely observed* if its cardinality is 1, i.e.,

$$\#t_{ij} \;\; = \;\; 1 \tag{12}$$

This gives rise to the definition of a completely observed scene description:

Definition 3 In a *completely observed scene description*, all relations must be completely observed, i.e.,

$$\forall_{i,j} \; \#t_{ij} \;\; = \;\; 1 \tag{13}$$

Proposition 1 For a scene description in which each relation has exactly one relation value, an inferred relation t'''_{ij} is either $\emptyset$ or t_{ij}.

Proof: An inferred topological relation t'''_{ij} is the empty topological relation if (1) two or more compositions, $t''_{ik}; t''_{kj}$, have no common relation value(s) or (2) any t''_{ik} or t''_{kj} is $\emptyset$. If two or more compositions have no common relation value, then their combination (the intersection) is empty, which corresponds to the empty topological relation. In the second scenario, a topological relation is composed with an empty topological relation the result of which is empty as well. From the first constraint follows that the intersection of all compositions is then empty as well.

The only non-empty topological relation that can be inferred is t_{ij} itself, because the calculation of t'''_{ij} includes as one of the terms the composition of $t''_{ii}; t''_{ij}$, which is a composition with the identity relation. Thus one of the terms of the intersection of the compositions is always t_{ij}. The intersection of all relevant compositions is then only non-empty if all other compositions contain t_{ij} as well. □

A topological relation t_{ij} is *inconsistent within a completely observed scene description* if the observed topological relation does not coincide with the inferred topological relation t'''_{ij} (Equation 11), i.e.,

$$t'''_{ij} \quad = \quad \emptyset \tag{14}$$

Conversely, a topological relation t_{ij} is *consistent within a completely observed scene description* if it is not inconsistent, i.e.,

$$t'''_{ij} \quad \neq \quad \emptyset \tag{15}$$

With the two possible relation values $\emptyset$ and t_{ij} for the induced topological relation, the definitions of topological consistency and inconsistency imply that in order to be consistent, a topological relation t_{ij} must coincide with the induced relation t'''_{ij}, i.e.,

$$t'''_{ij} \quad = \quad t_{ij} \tag{16}$$

A particular property of the set of eight binary topological relations allows for an abbreviated evaluation process of topological consistency of an incompletely observed scene description. In order to conform with the node consistency, it is necessary for all relations, $r\ (i, i)$, that the implied relation $r'''\ (i, i)$ contains *equal*, i.e.,

$$\forall_i \quad \textit{equal} \ \in \ \bigcap_{k=A}^{N} t_{ik}; t_{ki} \tag{17}$$

To fulfill this constraint, each composition $t_{ik}; t_{ki}$ must contain *equal*. Table 1 reveals that the only compositions that result in the identity relation are those of pairs of converse relations, which means that *equal* can only be inferred if t_{ik} and t_{ki} are converse. Thus, t_{ik} must be equal to $\breve{t}_{ki}$, which is the same condition as the arc consistency constraint. Thus, *in lieu* of evaluating the topological consistency for nodes, arcs, and paths, it is sufficient to guarantee that the constraint network of the topological relations is node- and path-consistent; therefore, Equation 11 can be immediately expressed in terms of a node consistent network.

$$\forall_{ij} \quad t'''_{ij} \quad := \quad \bigcap_{k=A}^{N} (t'_{ik}\ ;\ t'_{kj}) \tag{18}$$

6 REASONING ABOUT INCOMPLETELY OBSERVED SCENE DESCRIPTIONS

So far, it was assumed that the set of topological relations to be investigated for consistency is complete. This section will lift this limitation and allow a scene description to be *incompletely observed*. This generalization is crucial when reasoning about topology in geographic space, where incomplete observations are a common setting and information is frequently obtained by inference rather than observation (Chase and Chi,1981). To distinguish between a given set of topological relations that describe a scene and the information inferred by the consistency constraints, the respective terms *incompletely observed* and *incomplete* will be used. A topological relation with the highest possible degree of incompleteness corresponds to the universal topological relation $\mathcal{U}$ and will be called *unknown*.

Generally, incomplete observations are considered to provide less information than complete observations. Despite a higher degree of uncertainty about the actual value, it may be possible to infer more precise information, exploiting the knowledge about the correlation of the observations. New information may be an assessment of the consistency or the (partial) completion of an incompletely observed scene description. Incompletely observed scene descriptions give rise to such interesting questions as, "Can the unknown relations be inferred?", "Are all relations necessary?" and, "Is the given (incomplete) description consistent?" The latter assessment should be possible independent of whether the scene description can be completed or not.

Definition 4 A relation is *incompletely observed* if its cardinality is greater than 1, i.e.,

$$\#t_{ij} \quad \geq \quad 1 \tag{19}$$

Incomplete information about a topological relation may be expressed as a disjunction of several binary topological relations. Such disjunctions of possible values have been also termed *OR-objects* (Imielinski and Vadaparty, 1989) and *p-domains* (for domains of "possible" values) (Morrissey, 1990).

Example 10 The interiors of two objects, A and B, are separated, without any particular information about their boundaries. Figure 1 shows that there are two different topological relations, disjoint and meet, with empty interior intersections; therefore, the topological relation $r\ (A, B)$ is incompletely observed. It can be described as *disjoint or meet*. □

Example 11 Given two spatial regions, A and B, without any explicit information about their topological relation; however, it is known that A is considerably larger than B. This restricts the set of possible topological relations between A and B and implies that their topological relation is either *disjoint* or *meet* or *overlap* or *covers* or *contains* (Egenhofer and Al-Taha, 1992). □

An incompletely observed scene description is described by a set of topological relations in which for at least one ordered pair of objects, i and j, there is more than one binary topological relation.

Definition 5 An *incompletely observed scene description* has at least one incompletely observed relation, i.e.,

$$\exists_{i,j} \ \#t_{ij} \quad \geq \quad 1 \tag{20}$$

Figure 6 shows a network of an incompletely observed scene description. Incompletely observed topological relations are depicted by multiple edges between the same two nodes pointing in the same direction, whereas unknown relations are omitted in the graph.

The constraints about node consistency and path consistency hold for incompletely observed scene descriptions as well, because the composition distributes over disjunctions (Maddux, 1990); therefore, the basics of the inference mechanisms for the inconsistency and consistency of completely observed topological relations (Equations 14 and 15) apply immediately as a computational model for the evaluation of the consistency of incompletely observed topological relations; however, the underlying process of determining a path-consistent scene description (Equation 11) requires

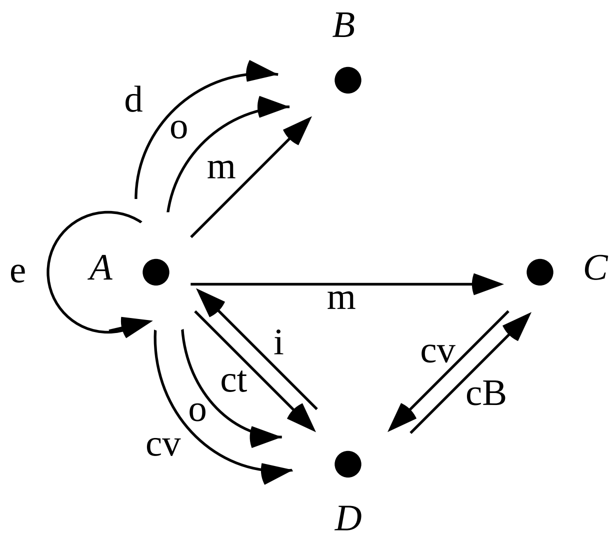

Figure 6: The constraint network for an incompletely observed scene.

some modification. An inferred topological relation may contribute to the inference of other topological relations in the same scene description and, therefore, may be used as additional knowledge in a new round of constructing a consistent scene description. Clearly, this inference process has to iterate until the induced scene description becomes stable, i.e., no new relations have been inferred.

Example 12 Given four spatial regions, A, B, C, and D and their topological relations A inside B, B inside C, and C inside D. All other relations are unknown, i.e., universal. The construction of the node-consistent network (Equation 9a and 9b) adds knowledge that each relation is equal to itself, and the arc-consistent network infers that B contains A, C contains B, and D contains C. The construction of a path-consistent network infers A inside C, B inside D, and their converse relations C contains A and D contains B. With this extended knowledge of the topology of the scene description it becomes possible to infer also A inside D, and its converse, D contains A. □

The major question is, "Over which steps has the algorithm to iterate?" First, it is sufficient to apply the node consistency only once at the beginning of the evaluation process, because it does not rely on new information derived from applying Equation 11. Second, since $r_i; r_j$ is the same as $\breve{r}_j; \breve{r}_i$, each iteration of constructing a path-consistent network implies that the network is also arc-consistent. Therefore, it is sufficient to iterate over the step constructing a path-consistent network. Equation 21 shows the modified algorithm for path consistency (Equation 11) for incompletely observed scene descriptions.

$$\begin{array}{ll} & \forall_{i,j} \;\; t'''_{ij} := t''_{ij} \\ \texttt{REPEAT} & \forall_{i,j} \;\; t''_{ij} := t'''_{ij} \\ & \forall_{i,j} \;\; t'''_{ij} := \bigcap_{k=A}^{N} t''_{ik} ; t''_{kj} \\ \texttt{UNTIL} & \forall_{i,j} \;\; t''_{ij} = t'''_{ij} \end{array} \tag{21}$$

Proposition 2 In an incompletely observed scene description, the induced relation t'''_{ij} can be $\emptyset$, t_{ij}, or a true subset of t_{ij}.

Proof: The first two scenarios follow immediately from the inference in a completely observed scene description (Proposition 1) and because t'_{ii} is also *equal* in an incompletely observed scene description. The third scenario may happen when the composition of $t''_{ii}; t''_{ij}$ has a subset in common with all other relevant compositions, but there is at least one relation value in t'''_{ij} that is not in one of the other compositions.

No other values for the induced relation are possible. The summary of the three possible scenarios is that $t'''_{ij} \subseteq t_{ij}$ or $t'''_{ij} = \emptyset$. A forth scenario could only exist if there was some t'''_{ij} such that $t'''_{ij} \cup t_{ij} \not\supset t_{ij}$; however, this is impossible, because $t''_{ii}; t''_{ij}$ is always t'''_{ij} and through the intersections with the other relevant compositions, it can be only restricted, not expanded. Thus, the largest possible t'''_{ij} is t_{ij} itself. □

This leads to general definitions of topological consistency and inconsistency, independent of whether the scene description is completely or incompletely observed.

Definition 6 A scene is *topologically inconsistent* if any induced relation is inconsistent, i.e.,

$$\exists_{i,j} \quad t'''_{ij} \quad = \quad \emptyset \tag{22}$$

Definition 7 For an entire scene to be *topologically consistent*, each induced relation must be consistent, i.e.,

$$\forall_{i,j} \quad \#t'''_{ij} \quad \geq \quad 1 \tag{23}$$

Note that an unknown scene (with all relations being universal) is topologically consistent.

For consistent, incompletely observed relations, a number of further properties can be formalized in terms of the inferred relation.

Definition 8 A consistent, incompletely observed relation is *determined* if the induced relation t'''_{ij} is unique, i.e.,

$$\#t'''_{ij} \quad = \quad 1$$

Example 13 In a scene with four spatial regions A, B, C, and D and the four observed topological relations *contains* (A, B), *meet* (B, C), *disjoint* (A, D), and *overlap* (D, C), the relationship between A and C can be uniquely inferred. Though both compositions, *contains (A, B)* ; *meet (B, C)* and *disjoint (A, D)* ; *overlap (D, C)*, are underdetermined, the intersection of the two compositions leads to a unique result:

$$\begin{aligned} \text{contains; meet} \;\cap\; \text{disjoint; overlap} \quad &= \quad (\text{contains} \vee \text{covers} \vee \text{overlap}) \;\cap \\ &\quad\;\; (\text{disjoint} \vee \text{meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap}) \\ &= \quad \text{overlap} \end{aligned}$$

□

Definition 9 A scene is *topologically determined* if the cardinality of each inferred relation t'''_{ij} is 1, i.e.,

$$\forall_{i,j} \quad \#t'''_{ij} \quad = \quad 1 \tag{24}$$

A topologically determined scene must be consistent as well, because if all t_{ij}'s are unique their cardinality is also greater than or equal to 1.

The intersections of the compositions of topological relations do not always result in singletons and it may be unclear for certain configurations, what value an unspecified topological relation may take.

Definition 10 An unknown topological relation t_{ij} is *topologically underdetermined* if the induced topological relation t'''_{ij} is underdetermined, i.e.,

$$\#t'''_{ij} > 1$$

Example 14 Given four spatial regions A, B, C, and D and the four topological relations *overlap* (A, D), *overlap* (B, D), *inside* (C, B), and *disjoint* (C, A). The topological relation t (C, D) cannot be inferred uniquely, because*:*

$$\begin{aligned} \text{disjoint; overlap} \ \cap \ \text{inside; overlap} &= (\text{disjoint} \vee \text{meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap}) \ \cap \\ &\quad (\text{disjoint} \vee \text{meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap}) \\ &= \text{disjoint} \vee \text{meet} \vee \text{inside} \vee \text{coveredBy} \vee \text{overlap} \end{aligned}$$

□

Definition 11 A scene is *topologically underdetermined* if it has no self-contradictions and the cardinality of any t'''_{ij} is greater than 1, i.e.,

$$\begin{aligned} \nexists_{i,j} \ \#t'''_{ij} &= 0 \quad \text{and} \\ \exists_{i,j} \ \#t'''_{ij} &> 1 \end{aligned} \tag{25}$$

A weaker notion of a determined relation is a relation that can be enhanced.

Definition 12 An incompletely observed relation can be *enhanced* if its inferred relation reduces the number of alternatives, without making the relation determined or inconsistent, i.e.,

$$\#t_{ij} > \#t'''_{ij} > 1$$

Example 15 Given three spatial regions, A, B, and C. It is know that B is smaller than C, which implies that their topological relations is *disjoint*, *meet*, *coveredBy*, *inside*, or *overlap* (Egenhofer and Al-Taha, 1992). Furthermore, it is known that B covers A and A meets C. Node consistency implies that t''_{BB} and t''_{CC} are equal. The relation between B and C can be enhanced, because

$$\begin{aligned} t'''_{BC} &= t''_{BA}; t''_{AC} \ \cap \ t''_{BB}; t''_{BC} \ \cap \ t''_{BC}; t_{CC} \\ &= \text{covers; meet} \ \cap \ \text{equal; (disjoint} \vee \text{meet} \vee \text{coveredBy} \vee \text{inside} \vee \text{overlap)} \ \cap \\ &\quad (\text{disjoint} \vee \text{meet} \vee \text{coveredBy} \vee \text{inside} \vee \text{overlap}); \text{equal} \\ &= (\text{meet} \vee \text{contains} \vee \text{covers} \vee \text{overlap}) \ \cap \\ &\quad (\text{disjoint} \vee \text{meet} \vee \text{coveredBy} \vee \text{inside} \vee \text{overlap}) \ \cap \\ &\quad (\text{disjoint} \vee \text{meet} \vee \text{coveredBy} \vee \text{inside} \vee \text{overlap}) \\ &= \text{meet} \vee \text{overlap} \end{aligned}$$

□

7 CONCLUSIONS

A method was described to evaluate whether or not a set of binary topological relations between spatial regions describes a scene consistently and without redundancy. It was based on a purely symbolic representation of spatial predicates without a need to model objects in a particular topological data structure. The computational model was based on a closed set of mutually exclusive topological relations between spatial regions and used the algebraic properties of the relations, primarily the converseness and the composition. By applying these properties as constraints among the topological relations of a scene description, the concepts of consistency and inconsistency were defined. For incompletely observed scenes, the consistency constraints also induce refined relations, up to the degree of completing the topology of a scene without adding additional information.

Although the discussions focused on point sets, any spatial data model that is a valid implementation of the point set concept may be used to implement the ideas discussed (Egenhofer and Herring 1991, Frank, 1992). Earlier work already reported about the application of the 4-intersection to modeling topological relations between simplicial complexes (Egenhofer 1989) and its implementation in commercial GISs for cells (Herring, 1991). GISs that record explicitly topology and solve topological conflicts based on the topology recorded, not by deriving it from such metric information as coordinates, provide a consistent and robust implementation. Otherwise, the consistency checks discussed here apply to detecting both inconsistent data and inconsistent geometry implementations.

The method has applications in a variety of fields such as data fusion to integrate multiple observations and spatial reasoning to infer new information. For geographic databases, it also applies to spatial query processing and optimization (Egenhofer, 1993).

The investigations showed that a scene of n objects needs less than all n^2 binary topological relations to be determined. In order to further optimize the processing of such queries, it is necessary to find the set of relations that predict the least cost of processing. In a first approximation, this may be considered the smallest set of topological relations if all relations have the same processing cost. The results of the present investigations suggest that such a minimal set contains somewhere between $(n-1)$ and $(n^2-n)/2$ relations (Egenhofer and Sharma, 1992), because node consistency eliminates n relations and arc consistency further reduces this set by half. On the other hand, at least $(n-1)$ relations are necessary to determine sufficiently a scene—less relations would definitely leave the topological relation to some object undetermined. The actual number of the smallest set of topological relations depends on the particular configuration. For example, a scene in which A meets B, B contains C, and A is disjoint from C, needs only the first two relations, because the third is induced by the others. On the other hand, if the relation between B and C was changed to overlap, all three relations would be necessary to determine the scene. While an exhaustive backtracking algorithm, removing one relation after another until the scene is topologically underdetermined, will identify the smallest set of relations, it is not the most efficient way. Ongoing research focuses on a more efficient algorithm, taking into consideration the particular properties of the topological relations.

In order to make this method of evaluating consistency really useful, it is necessary to extend the consistency analysis beyond topological relations. Similar relation algebras have to be developed for other spatial relations over spatial regions and integrated with the present method. Such algebras exist for point objects, describing cardinal directions (Frank, 1992, Freksa1992) and combinations of cardinal directions with approximate distances (Hernández, 1991), and similar formalisms for extended spatial objects are urgently needed.

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