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Modeling Conceptual Neighborhoods of Topological Line-Region Relations*

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Abstract

Based on the 9-intersection for binary topological relations, two models of conceptual neighborhoods among topological relations between a line and a region are developed. The *snapshot model* derives the neighborhoods by comparing pairs of topological relations and selects neighbors based on least noticeable differences, whereas the *smooth-transition* model develops neighborhoods based on the knowledge of the deformations that may change a topological relation. The resulting similarity diagrams show some differences, which were compared with the results from tests in which human subjects were asked to organized line-region relations into groups of similar relations. The groupings the subjects made indicate that the smooth-transition model captures more important aspects of the similarity of topological line-region relations than the snapshot model.

1. Introduction

The study of *spatial relations* aims at gaining a better understanding of the way people use them in everyday life—how they think about space and the relations among spatial objects, and how they communicate about them—and at developing methods suitable for the implementation in information systems. Over the last few years, this field has received increasing attention. Areas such as cognitive science (Lakoff and Johnson 1980) and linguistics (Talmy 1983; Herskovits 1986) have considerably influenced its advancement (Mark 1993). More recently, such investigations of spatial relations have demonstrated enormous practical relevance, for instance in the design and use of geographic information systems (Abler 1987; Mark and Frank 1991; Frank and Campari 1993).

This paper investigates properties of topological relations, i.e., spatial relations that are preserved under continuous transformations. We focus on the topological relations that result from a recent categorization (Egenhofer and Franzosa 1991; Egenhofer and Herring 1991). This model is very popular in the area of geographic information systems (GISs) as it has been applied to describe more detailed spatial relations (Herring 1991; Pigot 1991; Hazelton *et al.* 1992) and used for a number of applications in spatial query languages (Svensson and Zhexue 1991; Hadzilacos and Tryfona 1992).

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The model for topological relations used in this paper distinguishes 19 different topological relations between a line and a region in $\mathbb{R}^2$. Examples are such situations as a line's closure being a subset of the region's interior (Figure 1a), a line's closure as a subset of a region's closure (Figure 1b), the line outside of the region such that the line's interior intersects with the region's boundary (Figure 1c), and the line outside of the region such that only the line's boundaries, but not its interior, intersects with the region's boundary (Figure 1d). Some of these relations may be thought of being closer (or more *similar*) to each other than others. For example, 1a and 1b are fairly similar—the only difference is the location of one of the two endpoints, either in the region's interior or in its boundary. Also 1c and 1d are conceptually similar—the lines' interiors are both outside of the region's interior—however, the degree of similarity is considerably lower than the similarity between 1a and 1b.

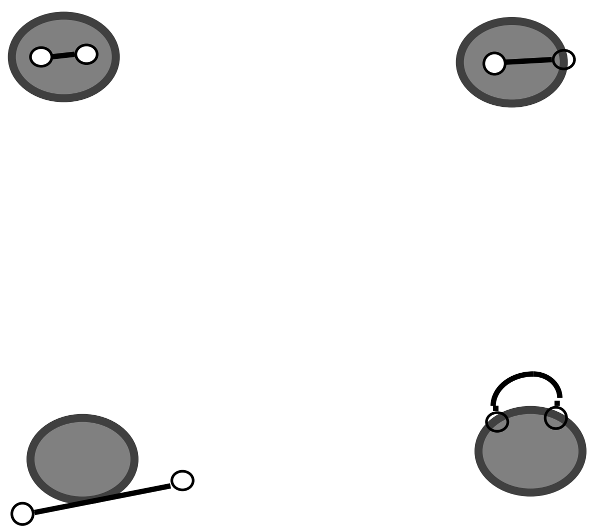

Figure 1: Two pairs of similar spatial relations.

The goal is to design a computational model that determines for each topological relation those relations that are conceptually closest to it. This model will be helpful in grouping topological relations according to their similarities, a task that is critical for the selection of appropriate terminology when people communicate with information systems about specific spatial configurations. The novel approach in this paper is the grouping of topological information, usually thought of as values on a nominal scale (Stevens 1946). The grouping implies a partial order over topological relations. This formal approach is complementary to human subject tests about the use of spatial relations in natural language (Mark and Egenhofer 1992). Conceptual neighbors of relations have been studied before; however, such models were based on visual analysis, rather than formal methods, and applied only to relatively simple objects such as 1-dimensional (temporal) intervals (Freksa 1992).

Following a brief summary of the model for topological relations (Section 2), this Technical Note focuses on the comparison of two different similarity models: The first model compares two snapshots of a line-region relation without any knowledge about the potential processes (or transformations) that may have occurred and selects conceptual neighbors based on the least number of differences (Section 3). The second model derives the closest relations from smooth transitions (Section 4). Using data from human-subjects tests, the significance of the two models is assessed (Section 5).

2. 9-Intersection

A *spatial region* is a connected, homogeneously 2-dimensional cell. Its formal definition is based on point-set topology with open and closed sets (Alexandroff 1961). Interior (A°), boundary (∂A), and exterior (A^-) of a 2-dimensional point set A embedded in IR^2 are defined as usual. The definition of a *line* is based on 1-cells, i.e., the direct connections between two geometrically independent nodes. A line is a sequence of $1 \ldots n$ connected 1-cells such that they neither cross themselves nor form cycles. Nodes at which exactly one 1-cell ends will be referred to as the *boundary of the line*, or briefly *boundary*. Nodes that are an endpoint of more than one 1-cell are *interior nodes*. The *interior* of a line is the union of all interior nodes and all connections between the nodes. The *closure* of a line is the union of its interior and boundary. Finally, the *exterior* is the difference between the embedding space and the closure of the lines. We will call a sequence of 1-cells a *simple line* if it has exactly two boundary nodes. A *complex line* would have more than two boundary nodes (Egenhofer and Herring, 1991). Lines with less than two boundary nodes would include cycles, which are excluded by definition.

Interior, boundary, and exterior will be referred to as the *topological parts* of an object. Among the parts there is an *adjacency* such that:

$$\text{adjacent}\,(A^\circ) = \partial A \tag{1a}$$

$$\text{adjacent}\,(\partial A) = A^\circ \text{ and } A^- \tag{1b}$$

$$\text{adjacent}\,(A^-) = \partial A \tag{1c}$$

The definition of binary topological relationships between a line L and a region R is based on the nine intersections of L's interior (L°), boundary (∂L), and exterior (L^-) with the interior (R°), boundary (∂R), and exterior (R^-) of R (Egenhofer and Herring 1991). A 3×3-matrix, M, called the *9-intersection*, concisely represents these criteria (Equation 2).

$$\mathsf{M} = \begin{pmatrix} L^\circ \cap R^\circ & L^\circ \cap \partial R & L^\circ \cap R^- \\ \partial L \cap R^\circ & \partial L \cap \partial R & \partial L \cap R^- \\ L^- \cap R^\circ & L^- \cap \partial R & L^- \cap R^- \end{pmatrix} \tag{2}$$

To refer to a particular intersection between two objects, the short form M [_, _] will be used, e.g., $\mathsf{M}\,[\partial, {}^-]$ to denote the value of the boundary-complement intersection.

By considering the values empty ( ) and non-empty (¬) for the 9-intersection, one can distinguish 512 binary topological relationships. The actual number of relationships that can be realized between two spatial objects embedded in a particular space, depends on the topological properties of the objects (Egenhofer and Franzosa 1991; Egenhofer and Herring 1991) and their codimensions, i.e., the difference between the dimension of the embedding space and the object (Egenhofer and Herring 1990; Herring 1991; Pigot 1991). For a simple line and a region in IR^2, it has been proven that only 19 different 9-intersections can be realized (Egenhofer and Herring 1991). This set is mutually exclusive and closed. Figure 2 shows the 9-intersections of the 19 binary topological relationships between a region and a line and depicts corresponding prototypical geometric interpretations.

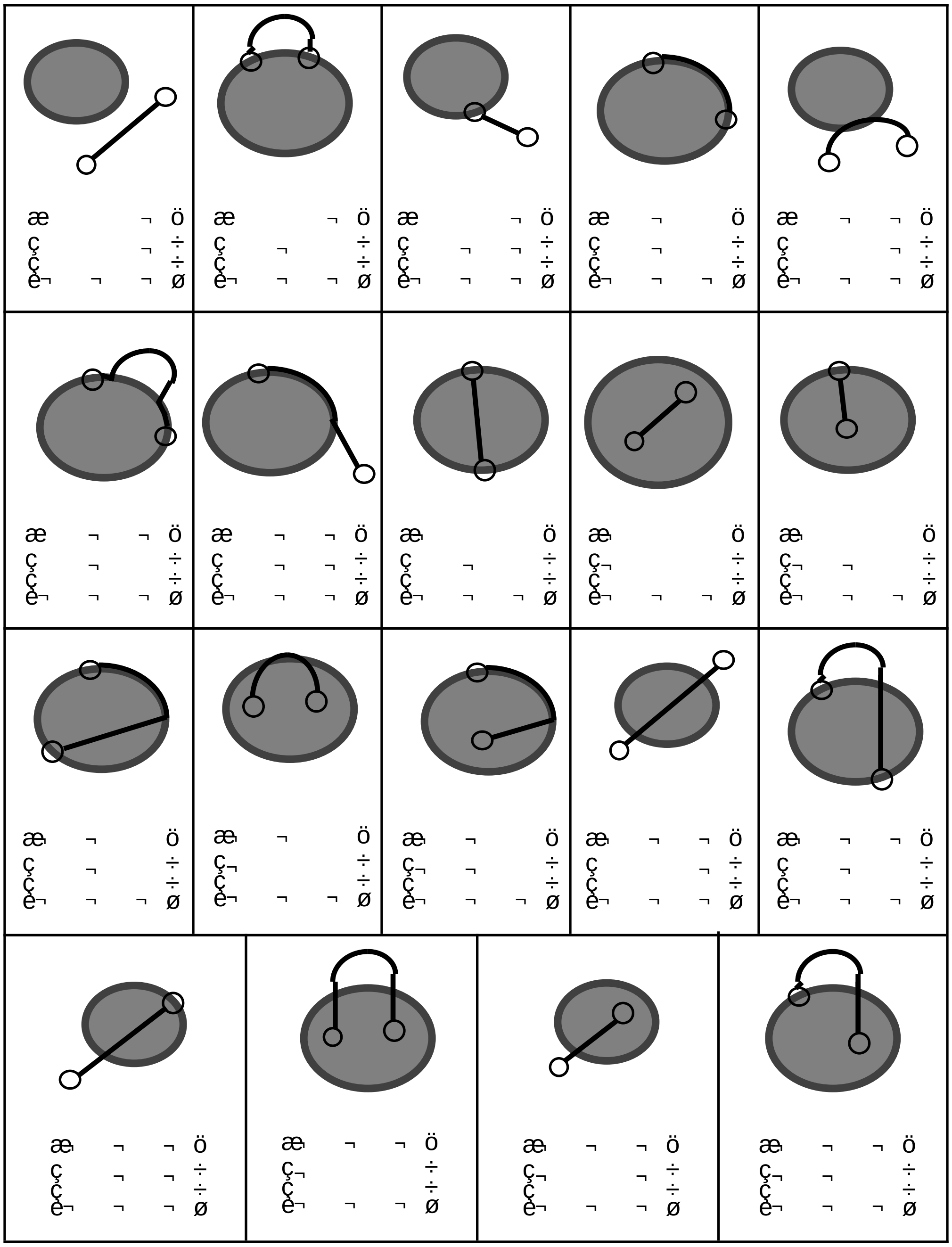

Figure 2: Geometric interpretations of the 19 line-region relations that can be realized from the 9-intersection.

The geometric interpretations reveal some intuitive similarities among the 19 relations. For example, the situation in which the line touches the region from outside at one of the end points is more similar to the situation in which the line is disjoint from the region than to the situations in which no part of the line is outside of the region.

3. Snapshot Model

The snapshot model compares two different topological relations without any knowledge about the potential transformations that may have caused their change. The comparison is based on the topological distance (Egenhofer and Al-Taha 1992), a measure for the remoteness of two different topological relations applied to their 9-intersections. We introduce the difference of empty/non-empty intersections, by mapping the values of empty and non-empty onto the integers 0 and 1, respectively, and applying then integer subtraction.

$$\begin{aligned} - \quad &= 0 \\ \neg \quad - \; \neg &= 0 \\ - \; \neg &= -1 \\ \neg \quad - \quad &= 1 \end{aligned} \qquad (3)$$

The *topological distance* between two topological relationships, r_A and r_B, is the count of differences of the empty/non-empty entries of corresponding elements in the 9-intersections (Equation 4).

$$T_{r_A, r_B} = \sum_{i=\circ}^{-} \sum_{j=\circ}^{-} |M_A [i, j] - M_B [i, j]| \qquad (4)$$

When calculated for each combination of line-region relations, the topological distance forms a 19×19 matrix of distances between pairs of topological relations. The shortest non-zero distance is 1 (topological distance 0 applies only between a relation and itself). For each relation r_I , the relations $r_{A \ldots N}$ with the shortest, non-zero topological distance are considered r_I's conceptual neighbors. They can be represented in a planar graph in which each relation is depicted as node and conceptual neighbors are the links between the nodes (Figure 3).

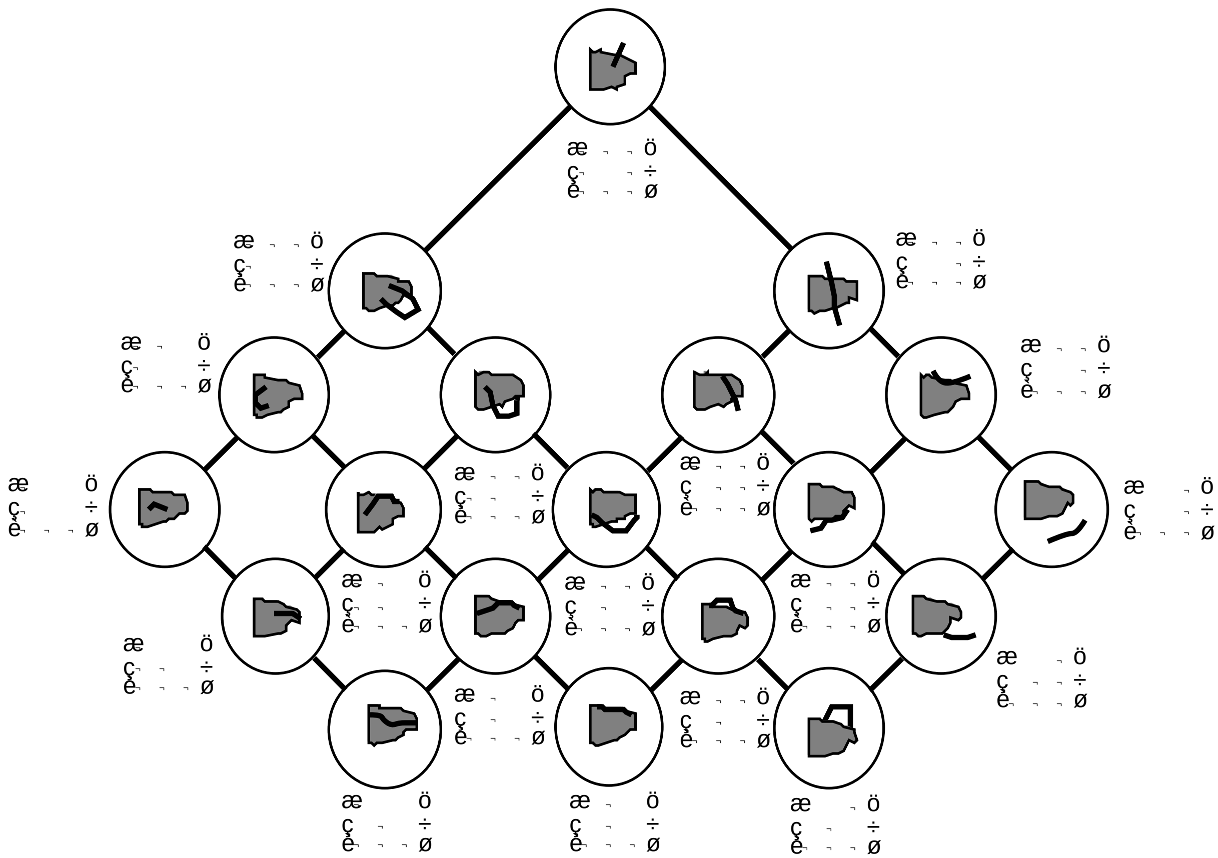

Figure 3: The conceptual neighborhoods derived from the snapshot model.

Each relation is a conceptual neighbor of at least two, and at most four other relations. The graph is symmetric with respect to the center column and its 9-intersections on the left hand are mirror images of the ones on the right hand. On the left-hand side are all relations in which the line is primarily in the region, while on the right-hand side are the relations in which the line is primarily outside. The intermediate cases, partially inside, partially outside, are toward the center of the diagram. Finally, the three symmetric relations are located in the center column of the diagram.

4. Smooth Transitions

The second model we will investigate is based on the concept of *smooth transitions*. In this model, two relations are conceptual neighbors if there exists a smooth transition from one to the other. By a smooth transition we understand an infinitesimally small deformation that changes the topological relationship. For lines and regions, such changes may be thought of as (1) pulling at the end of a line and (2) pushing the line's interior from one part of the region into an adjacent part . (Alternatively, the region's boundary could be moved among the parts of the line.) In terms of the 9-intersection, a smooth transition means that an intersection or its adjacent intersection (Equation 1) gets changed from empty to non-empty, or reverse. First, we introduce the *extent* of a part i, denoted by $\# M[i,\_]$, as the number of non-empty intersection between i and the three parts of the second object. For example, if the line's interior lies completely in the interior of the region, then its extent is 1 as it intersects with only one part of the region. The extent of a line's interior with respect to a region is in the interval of 1…3, the extent of the line's boundary is either 1 (if both nodes a located in the same region part) or 2 (if the nodes are located in different parts of the region), and the extent of a line’s exterior is always 3.

Based on this notion, the smooth transitions are formalized as follows:

(1) Moving a line’s boundary node from a region-part into an adjacent part of the region:

- If the line’s two boundaries intersect with the same region part then extend the intersection to either of the adjacent region parts.

$$\# M[\partial,\_] = 1 \qquad i(M[\partial, i] = \neg \quad): \quad M_N[\partial, \text{adjacent}(i)] := \neg \tag{5}$$

  For example, in figure 4a, the boundary-boundary intersection is the only non-empty intersection with the line’s boundary; therefore, the non-empty intersection is extended to the adjacent boundary-interior intersection and its adjacent boundary-exterior intersection.

- If the line’s two boundaries intersect with two different region parts then move either intersection to the adjacent region part.

$$\# M[\partial,\_] = 2 \qquad i(M[\partial, i] = \neg \quad): \quad M_N[\partial, i] := \quad \textit{and} \quad M_N[\partial, \text{adjacent}(i)] := \neg \tag{6}$$

  For example, in figure 4b, the line’s boundary intersects with the region’s boundary and interior; therefore, the non-empty boundary-boundary intersection is moved to the adjacent (empty) boundary-exterior intersection such that the boundary-exterior intersection turns form empty to non-empty and the non-empty boundary-interior intersection is moved to the adjacent (non-empty) boundary-boundary intersection such that the boundary-interior intersection turns empty.

(2) Moving a line’s interior partially from a region-part into an adjacent part of the region:

- Extend the line’s interior-intersection to either of the adjacent region parts.

$$i(M[{}^\circ, i] = \neg \quad): \quad M_N[{}^\circ, \text{adjacent}(i)] := \neg \tag{7}$$

  For example, in Figure 4c, the line’s interior intersects with the region’s boundary; therefore, the non-empty interior-interior intersection is extended to its adjacent empty boundary-interior intersection and its adjacent non-empty boundary-exterior intersection.

- Reduce the line’s interior-intersection on either of the adjacent region parts.

$$\# M[{}^\circ,\_] = 2 \qquad i(M[{}^\circ, i] = \neg \quad): \quad M_N[{}^\circ, i] := \tag{8a}$$

$$\# M[{}^\circ,\_] = 3 \qquad i(i \neq \partial): \quad M_N[{}^\circ, i] := \tag{8b}$$

  For example, in Figure 4d, the line’s interior extends over the region’s boundary and exterior; therefore, the line’s interior-intersection is reduced, moving the line’s boundary entirely into the region’s boundary and exterior.

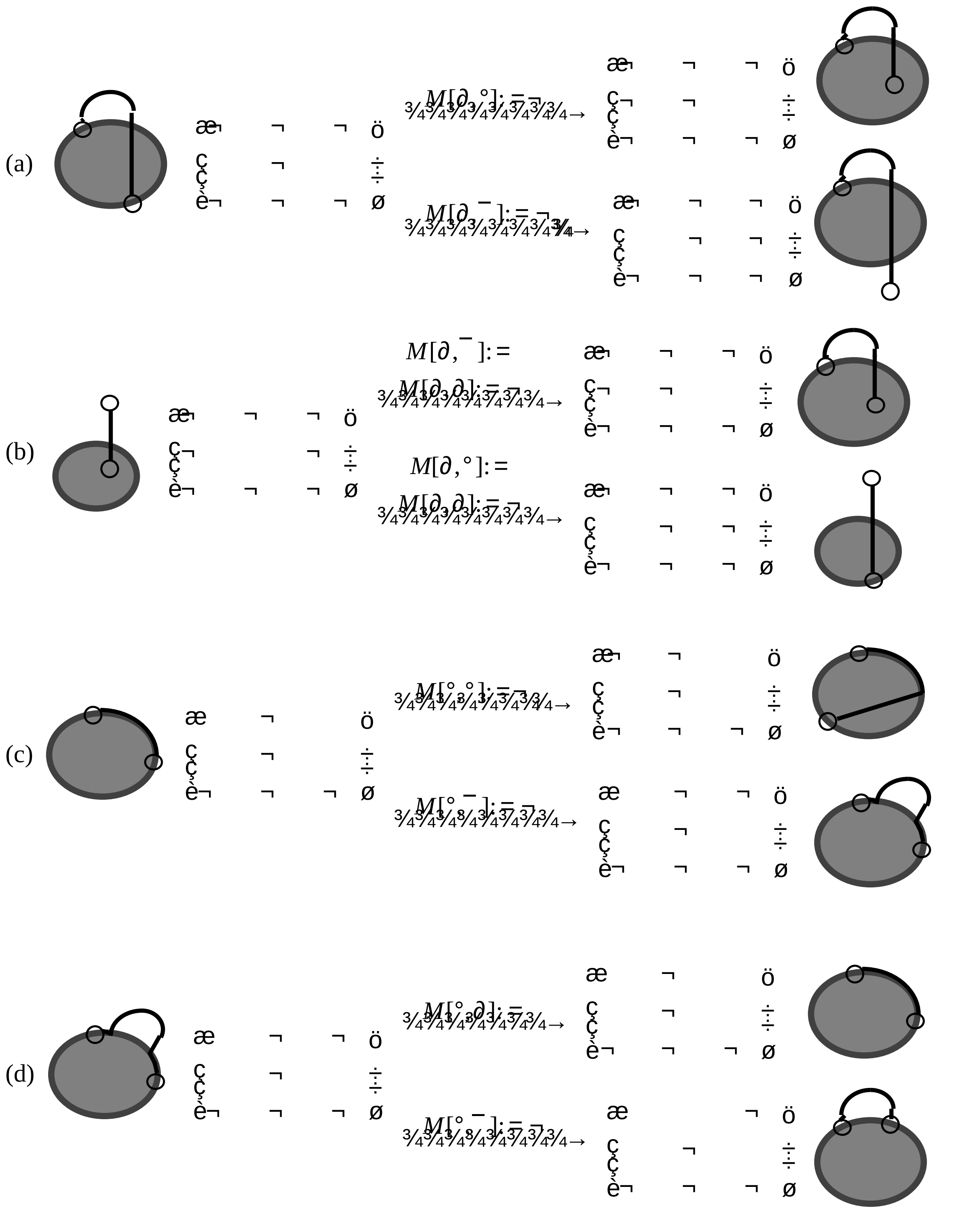

Figure 4: (a) Moving one boundary of a line into an adjacent part of the region, (b) either boundary into an adjacent region part, (c) the line's interior into an adjacent part of the region, and (d) the line's interior out of a part of the region.

The separate moves of the line's interior and boundaries (Equations 5-8) are atomic operations that do not account for some of the properties of the objects and their embedding space and, therefore, may generate inconsistent 9-intersections for configurations that cannot be realized.

In order to maintain connectivity among the line's boundaries and interior, it is necessary to assure the following consistency constraint:

- If the line's interior intersects with the regions interior *and* exterior, then the line's interior must also intersect with the region's boundary.

$$M[^\circ,^\circ] = \neg \quad \textit{and} \quad M[^\circ,^-] = \neg \qquad M[^\circ,\partial] := \neg \tag{9}$$

Likewise, in order to preserve the continuous-space property of IR^2, the following consistency constraint must be fulfilled:

- If the line's boundary intersects with the region's interior (exterior) then the line's interior must intersect with the region's interior (exterior) as well.

$$M[\partial,^\circ] = \neg \qquad M[^\circ,^\circ] := \neg \tag{10a}$$

$$M[\partial,^-] = \neg \qquad M[^\circ,^-] := \neg \tag{10b}$$

Equations (5-10) establish the smooth transitions for line-region relations. When applied to the 19 line-region relations, they provide the neighborhood graph shown in Figure 5. This graph resembles in most of its structure and properties the snapshot neighborhood graph (Figure 3). The differences between the two graphs are (1) the way in which conceptual neighbors are connected at the top and (2) the additional links that run across the smooth transition graph.

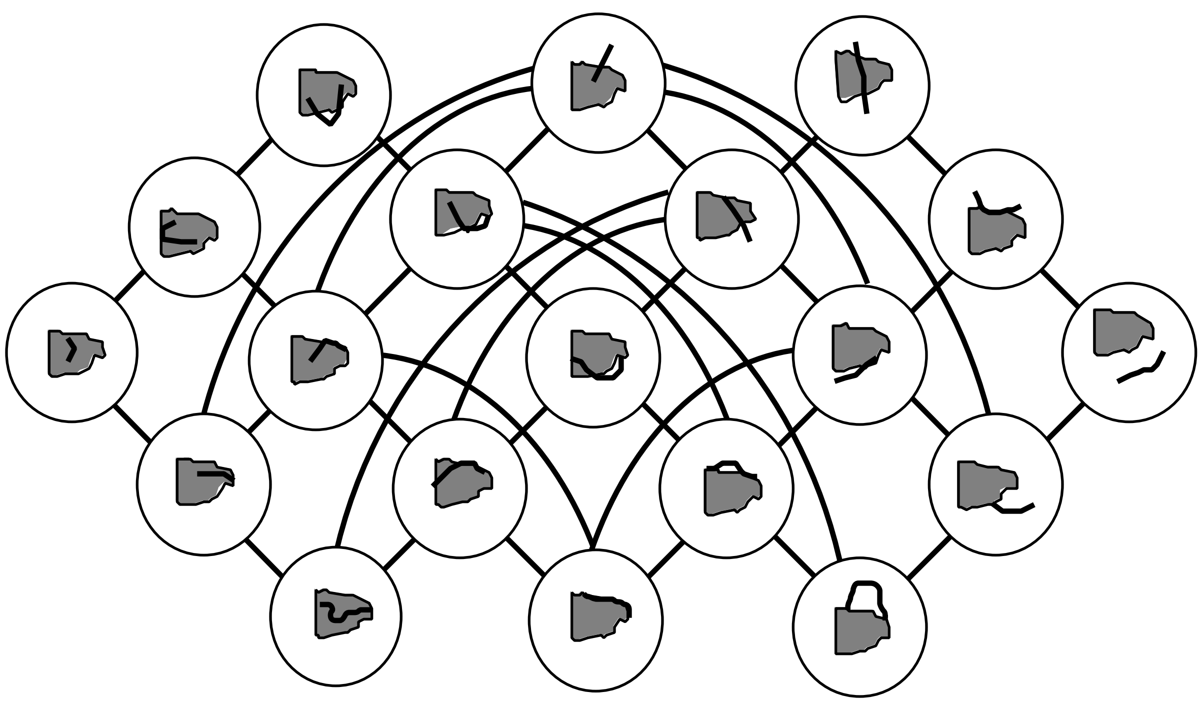

Figure 5: The conceptual neighborhoods derived from the smooth-transition model.

5. Comparison with human behavior

The 19 line-region relations of the 9-intersection give rise to 171 distinct pairs $(19 \times 18/2)$ of relations that can possibly be conceptual neighbors. Of these pairs, 26 are conceptual neighbors under both the snapshot model and the smooth-transition model; two relation pairs are conceptual neighbors by the snapshot model definition, but not the smooth transitions; twelve pairs are conceptual neighbors by smooth transitions, but not under the snapshot model; and the remaining 131 pairs are conceptual neighbors under neither model.

In an earlier study, we obtained human-subjects data on grouping of spatial relations (Mark and Egenhofer 1992, 1994), when 28 subjects performed a grouping task involving 38 diagrams, each of which showed a line and a region, said to be a road and a park, respectively. The parks were all the same size and shape, and there were 2 geometrically distinct placements of the road corresponding to each of the 19 topologically distinct relations. Since for each pair of relations there were exactly two examples, each spatial relation could be grouped as many as 112 times (4 pairs times 28 subjects) with each other relation. As a basis for comparison, the pairs *within* each relation distinguished by the 9-intersection were grouped by all 28 subjects for 7 of the 19 relations, and by at least 23 subjects (82%) for every relation. The maximum number of times that the stimuli in two different spatial relations were paired was 78 out of a maximum of 112 times (70%).

Within the context of different models for conceptual neighbors, it is particularly enlightening to analyze how the subjects formed groups of similar relations. The pairs that were neighbors by both snapshot and smooth-transition models were grouped from 0 to 78 times, with a mean of 33.6 (Figure 6).

- Those pairs that were neighbors for smooth transitions—but not snapshots—were grouped between 0 and 66 times, with a mean of 17.3 (15.4%).
- The two pairs that were snapshot neighbors—but not smooth transition neighbors—were grouped 10 and 16 times (mean = 13; 11.6%).

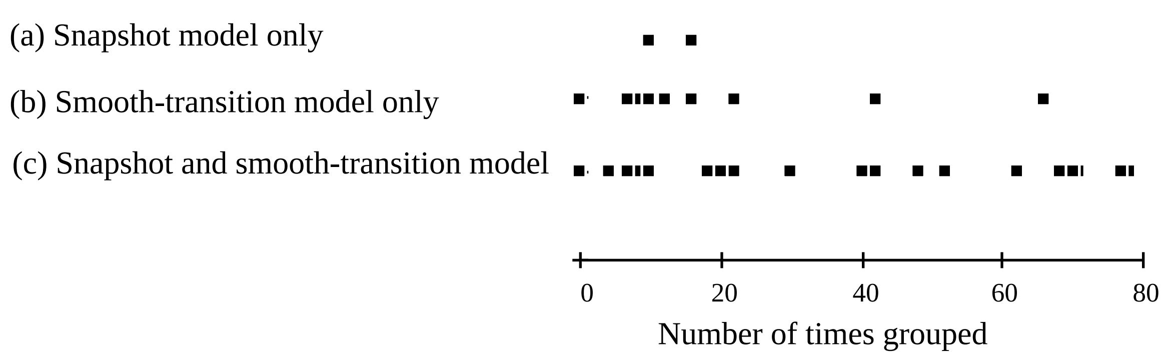

Figure 6: The number of times subjects grouped pairs of relations that are conceptual neighbors in (a) the snapshot model only, (b) the smooth-transition model only, and (c) in both models.

Perhaps most significant, however, is the fact that the 131 pairs that were neighbors by neither the snapshot model nor the smooth transitions were grouped an average of only 6.0 time by the subject (5.3% of the maximum). Sixty pairs were never grouped by any of the 28 subjects nor any of the four possible stimulus pairs. The most frequently-grouped pair in this category was 54 times (48%), but only 20 stimulus pairs with neither smooth transitions nor minimum snapshot difference were grouped 12 or more times (10% of the maximum).

6. Conclusions

Two formal models for the similarity of topological line-region relations led to almost identical conceptual-neighborhood diagrams. Tests with human subjects confirmed that the conceptual neighborhoods identified by the two models correspond largely to the way humans conceptualize similarity about spatial relations. The smooth-transition model represented the change process explicitly, whereas the snapshot model inferred change from topological differences. Since the majority of conceptual neighbors is the same in both diagrams, we conclude that the knowledge of a change process can be generally neglected when only considering topological similarity.

7. Acknowledgment

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