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On the Equivalence of Topological Relations*

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Abstract

The 4-intersection, a model for binary topological relations, is based on the intersections of the boundaries and interiors of two point sets in a topological space, considering the content invariant (i.e., emptiness/non-emptiness) of the intersections. If the 4-intersections of two pairs of point sets have different contents, then their topological relations are different as well; however, the reverse cannot be stated as there may be different topological relations that map onto a 4-intersection with the same content. This paper refines the model of empty/non-empty 4-intersections with further topological invariants to account for more details about topological relations. The invariants used are the dimension of the components, their types (touching, crossing, and different refinements of crossings), their relationships with respect to the exterior neighborhoods, and the sequence of the components. These invariants, applied to non-empty boundary-boundary intersections, comprise a classification invariant for binary topological relations between homogeneously 2-dimensional, connected point sets (disks) in the plane such that if two different 4-intersections with the necessary invariants are equal, then their topological relationships are identical. The model presented applies to processing GIS queries about whether or not two pairs of spatial objects have the same topological relation and gives rise to the formal definition of *topological similarity*.

1. Introduction

In a previous paper, we introduced a formal method to distinguish different topological relations between two point sets in a topological space (Egenhofer and Franzosa 1991). The model, called the *4-intersection*, is based on the four intersections between the boundaries and interiors of the two point sets and uses the categorization of the intersections according to their content (i.e., emptiness/non-emptiness). It applies immediately to such representations of 2-dimensional geographic objects as GT-polygons in the Spatial Data Transfer Standard (Fegeas *et al*. 1992), cells (Herring 1991) or simplicial complexes (Frank and Kuhn 1986; Egenhofer 1989; Egenhofer *et al.* 1989; Worboys 1992).

Over the last few years, the 4-intersection has enjoyed considerable popularity among GIS and database researchers. For example, Smith and Park (1992) showed how the point-set relations form a relation algebra. Svensson and Zhexue (1991) incorporated the model into a spatial-analysis query language. De Hoop and van Oosterom (1992) designed a topological query

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language around these operators. Herring (1991) discussed possible extensions of the model to cover topological relations between lines in IR^2. Hadzilacos and Tryfona (1992) and Clementini *et al.* (1993) investigated how the 4-intersection behaves when applied to model topological relations between regions, lines, and points in IR^2, and Pigot (1991) and Hazelton *et al*. (1992) applied the model to 3- and 4-dimensional objects. Likewise, designers of software for GISs and database management systems (DBMSs) have found considerable interest in the 4-intersection model and there are now implementations available in a commercial GIS (Herring 1991) and an extension of a commercial relational DBMS (Keighan 1993).

The sound model also enabled a number of advanced theoretical studies such as the formal derivation of the composition table for this set of relations (Egenhofer 1991), the design of comprehensive reasoning systems to detect inconsistencies in topological descriptions (Egenhofer and Sharma 1992), investigations of temporal changes of topological relations (Egenhofer and Al-Taha 1992), and a comparison of the similarities and dissimilarities of topological relations in a vector and raster model (Egenhofer and Sharma 1993). Egenhofer and Herring (1991) extended the principle of boundary and interior intersections with exterior intersections, which provides a generic model for topological relations involving *n*-dimensional objects embedded in higher-dimensional spaces. Parts of these extensions are now being tested with human subjects to identify how closely the formalism represents human cognition (Mark and Egenhofer 1992).

While the 4-intersection with the content invariant of empty and non-empty intersections has proven to be simple and thus attractive, a frequent concern has been that it is too generic, not being capable of distinguishing among a number of situations for which humans would have distinct mental images (Clementini *et al.* 1993; Herring 1991; de Hoop and van Oosterom 1992). This becomes most apparent if one considers the multitude of configurations that can be realized for the overlap specification. Figure 1 shows some examples, all of which have the same 4-intersection with empty/non-empty values, but display topologically distinct configurations. Other topological invariants than the content of the intersections may enable the distinction of more details about such relations. For example, the dimension of the intersection and the number of components of non-empty intersections may be used as additional topological invariants to find more details than with the content invariant (Egenhofer and Herring 1990; Egenhofer 1993). Likewise, to differentiate some topological relations between lines in IR^2, the type of the interior intersections (crossing or touching) has been suggested as a topological invariant (Herring 1991).

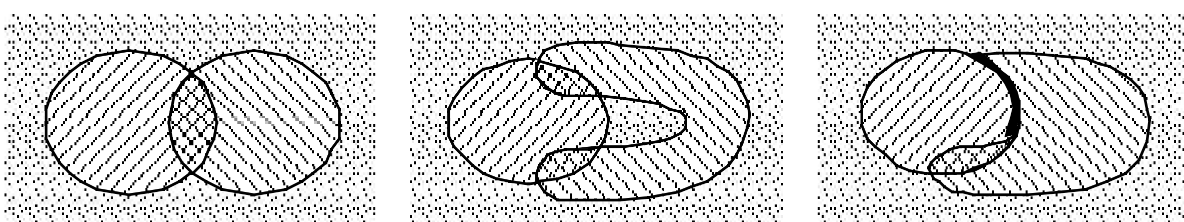

Figure 1: Three topologically distinct relations, all of which have the same 4-intersections.

This paper develops a comprehensive formalism to distinguish between such topologically distinct configurations whose empty/non-empty 4-intersection values are the same. It results in the definition of a mechanism to determine whether two or more topological relations between two 2-dimensional objects that are homeomorphic to disks in the plane are the same. This new method is a refinement of, not a replacement for, the previously introduced formalism with empty/non-empty intersections. Empty/non-empty intersections remain the most general measure to distinguish between different topological relations and the foremost measure for qualitative assessments. The distinction of emptiness and non-emptiness is also fully sufficient to define the equivalence of several of the point-set relations (equal, disjoint, inside, contains); however, other topological invariants can be used to describe more details about other relations, whenever this becomes necessary.

The practical motivation behind these theoretical investigations is the use of spatial relations as constraints in GIS query languages (Frank 1982; Egenhofer 1992). Usually, GISs do not record explicitly the spatial relations among spatial objects, but rather derive them during query processing from some spatial data model. Most GISs employ a set of separate methods (implementations) for each topological relationship, which requires an extension in the code whenever a user requests to use a new topological relation—an exception is MGE Dynamo, which uses a model for topological relations that is based on a derivative of the 4-intersection (Herring 1991). A comprehensive theoretical investigation of topological relations and their equivalence will help GIS designers in identifying a small and non-redundant set of operations, from which all possible combinations can be derived. It will further enable GIS designers to include into their query languages analytical operations to compare topological relations for equality and similarity.

For the remainder of this paper our investigations will focus on the topological relations between 2-disks—connected, homogeneously 2-dimensional point sets with a single boundary, also called GT-polygons in SDTS (Fegeas *et al.* 1992)—embedded in the Euclidean plane IR^2.

Section 2 briefly summarizes the formalism of the 4-intersection and the content invariant. Section 3 motivates the investigation of other topological invariants by discussing what topological differences the content invariant can detect and what differences it cannot. Section 4 introduces component invariants as finer criteria for non-empty intersections. Section 5 defines for disk-like objects a classifying invariant in terms of the content invariant of the 4-intersection, in combination with four component invariants, and proves the need for all four component invariants in the most general setting. Section 6 identifies the links and dependencies between particular content invariants and the classifying invariant. The conclusions in section 7 discuss the implications of the results and related questions that can be investigated in the future based on the present findings.

2. 4-Intersection and its content invariant

The usual concepts of point-set topology with open and closed sets are assumed (Alexandroff 1961; Munkres 1975). A *binary topological relation*, or just *topological relation*, is defined as a triple (A, B, X) where A and B are subsets of the topological space X. Given two topological spaces X, Y and two pairs of sets, A_X, B_X and A_Y, B_Y in X and Y, respectively, the topological relation between A_X and B_X in X is said to be *equivalent* to the topological relation between A_Y and B_Y in Y if there is a homeomorphism f: $X \rightarrow Y$ such that $f(A_X) = A_Y$ and $f(B_X) = B_Y$.A homeomorphism is a bijective bicontinuous function f from x to y.

The mechanics of distinguishing different topological relations are based on the point sets' interiors and boundaries. The *interior* of a set A, denoted by $A°$, is the union of all open sets in A. The closure of A, denoted by $\overline{A}$, is the intersection of all closed sets containing A. The exterior of A with respect to the embedding space IR^2, denoted by A^-, is the set of all points of IR^2 not contained in A. The *boundary* of A, denoted by ∂A, is the intersection of the closure of A and the closure of the exterior of A. Binary topological relations between two point sets, A and B, are examined via the four intersections of A's boundary (∂A) and interior ($A°$) with the boundary (∂B) and interior ($B°$) of B (Egenhofer and Franzosa 1991) and represented as a 4-tuple ($\partial A \cap \partial B$, $A° \cap B°$, $\partial A \cap B°$, $A° \cap \partial B$), called the *4-intersection*. By considering the values empty ( ) and non-empty (¬) for the 4-intersection, one can distinguish sixteen binary topological relations. Eight topological relations can be realized for two 2-disks that are embedded in IR^2 (Egenhofer and Franzosa 1991). This set provides a complete coverage for disk relations and is mutually exclusive, i.e., exactly one of these topological relations holds true between any two 2-disks in IR^2. We call them *disjoint*, *meet*, *equal*, *inside*, *contains*, *covers*, *coveredBy*, and *overlap* (Figure 2), though any other notation such as $R_0 \ldots R_7$ would do the same service.

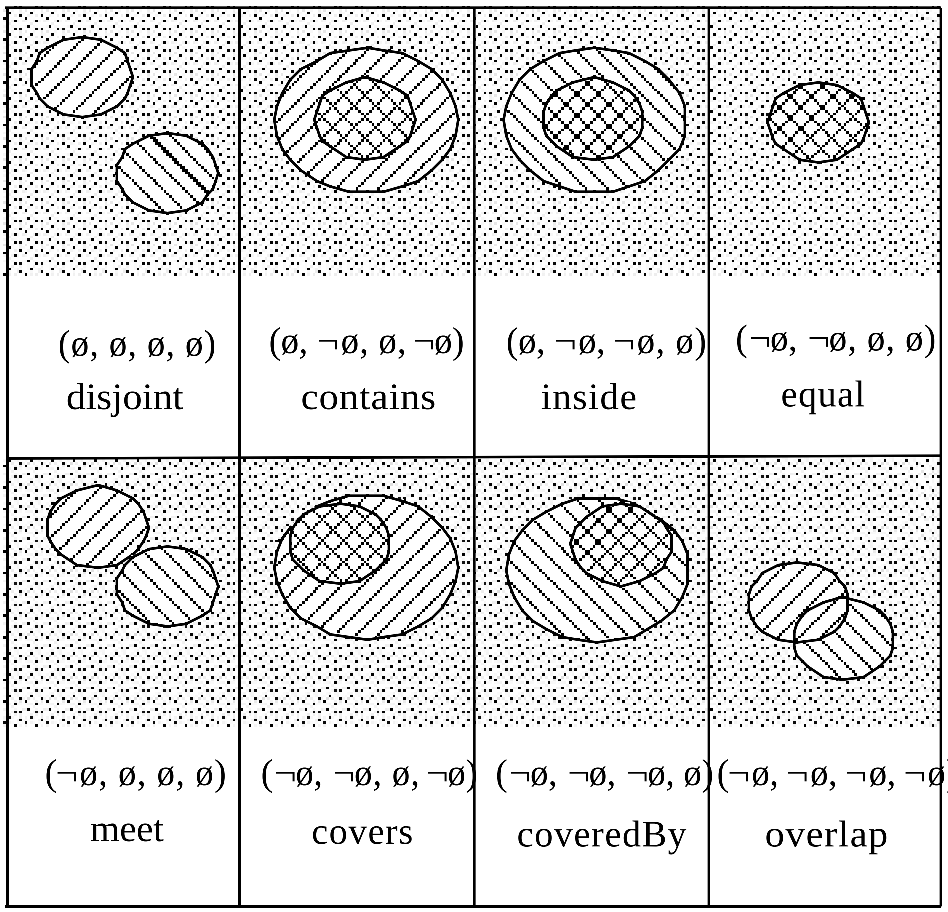

Figure 2: The eight topological relations that can be realized for two 2-disks that are embedded in IR^2.

3. Implications from the content invariant of 4-intersections

The distinction of empty and non-empty intersections is called the *content invariant*, a topological property of the 4-intersection. This section investigates the relationship between the content invariant and corresponding topological relations and analyzes what one can infer about the content of the 4-intersections if two topological relations are equivalent and what is implied about the topological relation if two pairs of 2-disks have the same content for their 4-intersections.

Let A_X, B_X, A_Y, and B_Y be 2-disks embedded in the topological spaces X and Y, respectively. The four sets $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, and $A^\circ \cap \partial B$ (and the associated values of empty or non-empty) depend on the topological relation between A and B in their corresponding spaces. It follows that if the topological relation between A_X and B_X in X is equivalent to the topological relation between A_Y and B_Y in Y, then the associated 4-intersection values are identical, but not necessarily the reverse. That is, it does not follow that if the associated 4-intersections are identical, then the topological relations are equivalent. Figures 3 and 4 show an example why this conclusion would be incorrect. In figure 3 the topological relation between A_X and B_X in X is equivalent to the topological relation between A_Y and B_Y in Y, and therefore, the corresponding 4-intersections are identical, i.e., equal to (¬ , ¬ , ¬ , ¬).

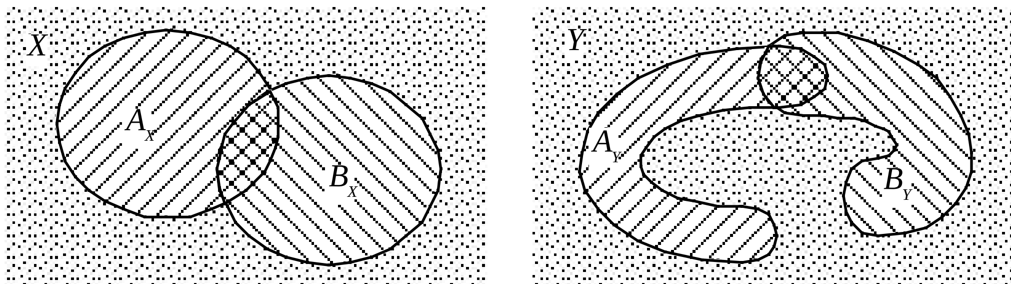

Figure 3: Two configurations whose 4-intersections are both (¬ , , ,) and whose topological relations are the same.

In figure 4 the corresponding 4-intersections are identical—again, equal to (¬ , ¬ , ¬ , ¬)—but the topological relation between A_X and B_X in X is not equivalent to the topological relation between A_Z and B_Z in Z

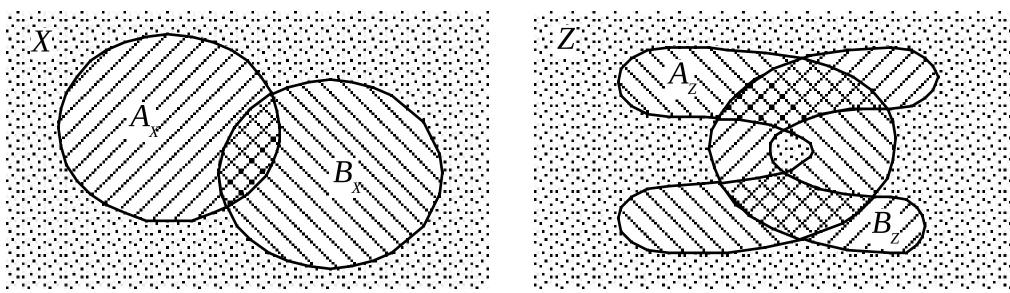

Figure 4: Two configurations whose 4-intersections are both (¬ , , ,), but whose topological relations are different.

It is the contrapositive statement that can be exploited in studying topological relations; that is, if the corresponding 4-intersections are not identical then the associated topological relations are not equivalent. For example, in figure 5 the 4-intersections are not the same, and therefore one can conclude that the corresponding topological relations are not equivalent.

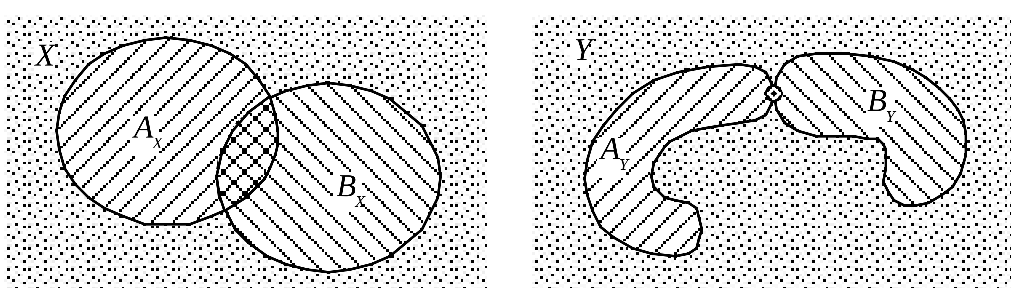

Figure 5: Two configurations with different 4-intersections, implying that their topological relations are different as well.

We will strengthen the information in the 4-intersection to obtain topological invariants that provide finer distinctions between topological relations. For example, if the information in the 4-

intersection consists of empty or non-empty, then the 4-intersections for the two topological relations in figure 6 are identical; however, if the information in the 4-intersection also includes the number of connected components of the corresponding intersection, the 4-intersections would be different. This follows from $\partial A_X \cap \partial B_X$ having one component, whereas $\partial A_Y \cap \partial B_Y$ has two components. From this difference one can conclude that the corresponding topological relations are not equivalent either. Similarly, in figure 7 if the information in the 4-intersection consists of empty or non-empty, then the 4-intersections for the two topological relations are identical; however, if the information in the 4-intersection also includes the dimension of the corresponding intersection the two 4-intersections would be different, because $\partial A_X \cap \partial B_X$ has dimension 0, while $\partial A_Y \cap \partial B_Y$ has dimension 1. Again, one can conclude from the differences in the 4-intersection that the corresponding topological relations are not equivalent.

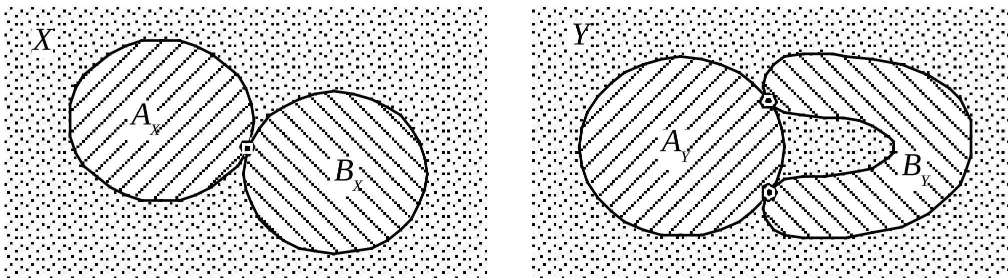

Figure 6: Two configurations that are distinguished by different numbers of connected components of the boundary-boundary intersections.

4. Component invariants of 4-intersections

To consider a broader class of invariants with which one can make finer distinctions between topological relations, we introduce the notion of a 4-intersection invariant. Let S represent either $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, or $A^\circ \cap \partial B$ for A and B in X.

Definition 1: A *4-intersection invariant* for the topological relation between A and B in X is a four-tuple ($I_{\partial A \cap \partial B}$, $I_{A^\circ \cap B^\circ}$, $I_{\partial A \cap B^\circ}$, $I_{A^\circ \cap \partial B}$) where each entry I_S is a (non-empty) set of properties of S and each such property is invariant under topological relation equivalence.

The choice of the invariants to use in a 4-intersection invariant in a particular situation can be made to distinguish best between the topological relations under consideration. For example, if the entries in the 4-intersection invariant represent the cardinality of the set of connected components of the corresponding intersection—number of connected components if there are finitely many—then in figure 6 the 4-intersection invariant for the topological relation between A_X and B_X in X is (1, 0, 0, 0), while that for the topological relation between A_Y and B_Y in Y is (2, 0, 0, 0), therefore, one can formally distinguish between the two topological relations. Alternatively, if the entries in the 4-intersection are either   if the corresponding intersection is empty or the topological dimension of the corresponding intersection if it is not empty, then in figure 7 the 4-intersection invariant for the topological relation between A_X and B_X in X is (0,   ,   ,   ), while that for the topological relation between A_Z and B_Z in Z is (1,   ,   ,   ), and again the two topological relations can be distinguished.

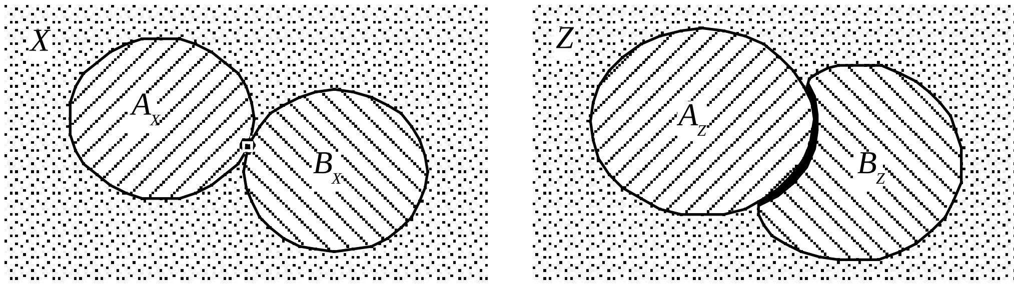

Figure 7: Two configurations that distinguish by different dimensions of the components of the boundary-boundary intersections.

4.1 Components

A non-empty 4-intersection may have several distinct components, each of which may be characterized by its own topological properties. Figure 8 shows two sets whose (non-empty) boundary-boundary intersection has three separate components, C_0, C_1, and C_2, with different topological properties. The definition of a component is based on the topological concept of connectedness. Let $Y \quad \underline{X}$. A *separation* of Y is a pair A, B of subsets of X such that $A \neq \quad$ and $B \neq \quad$; $A \quad B = Y$; and $\bar{A} \cap B = \quad$ and $A \cap \bar{B} = \quad$. If there exists a separation of Y then Y is said to be *disconnected*, otherwise Y is said to be *connected*. A *component* of Y is then a largest connected (non-empty) subset of Y. It is a topological invariant, i.e., under any topological transformation a component transforms to a component. It can neither disappear—turn into an empty set—nor can it merge with another component.

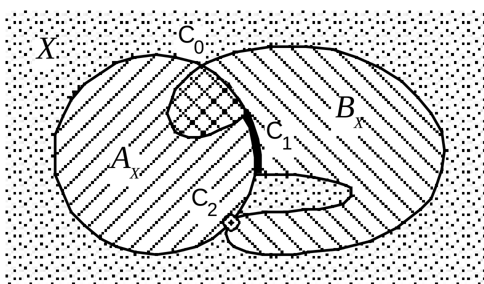

Figure 8: A topological relation with three boundary-boundary components, C_0, C_1, and C_2.

Within this setting, one can re-evaluate the nature of the content invariant of the 4-intersection. Obviously, there is a dependency between the content and the component invariant: an intersection is empty if and only if it has zero components, and an intersection is non-empty if and only if it has at least one component. Considerations about the components of empty intersections (i.e., empty sets) are useless; therefore, the following investigations will consider only components of non-empty intersections. Their topological invariants are called *component invariants*.

Analyzing the topological relations in figure 9 for components, we find that the boundary-boundary intersection between A_X and B_X in X has 6 components, the interior-interior intersection 2, the boundary-interior 2, and the interior-boundary 4. In the same figure, the relation between A_Y and B_Y in Y is topologically different from the one between A_X and B_X, however, both have the same number of components. Obviously, the pure counting of components is insufficient to determine whether two topological relations are identical or not. In order to determine whether two pairs of 2-disks have the same topological relation, it is necessary to consider topological invariants of each individual component as well as topological properties in the relationship between an intersection and all its components.

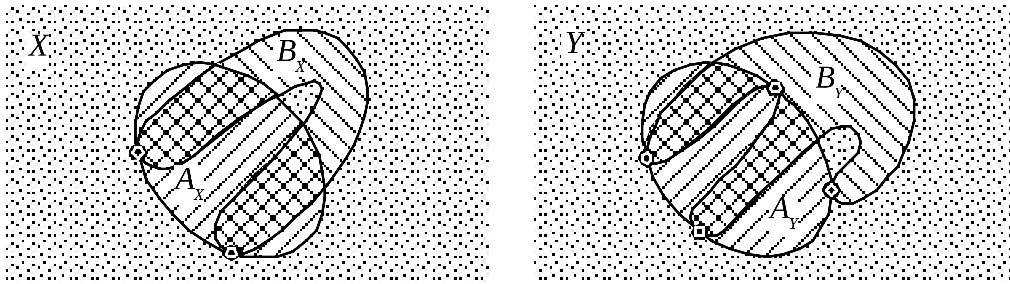

Figure 9: Two topologically distinct relations with the same number of boundary-boundary components.

Subsequently, the focus will be on the components of the boundary-boundary intersection and their topological invariants, though most of the discussions will apply to the components of the other three intersections as well. Actually, it will be shown later that for two 2-disks in the plane to be topologically equivalent, it is necessary and sufficient to consider a particular subset of topological invariants for the components of the boundary-boundary intersection only, because the combination of their properties implies the corresponding properties of the components of the other intersections.

4.2 Dimension of the components

The concept of *dimension of a component* needs naturally a definition of *dimension*. We adopt the usual definition from topology that is based on the dimension of the empty set being -1. The dimension of a space is then the smallest integer n such that every point of the space has arbitrarily small neighborhoods whose boundaries have dimension less than n. Thus, a point has dimension 0, a line is of dimension 1, the plane is of dimension 2, etc.

In analogy to the component invariant, there is a dependency between the content and the dimension invariant:

- an intersection is empty if and only if its dimension is -1, and
- an intersection is non-empty if and only if its dimension is greater than or equal to 0.

4.3 Boundary-boundary component sequence

Sequences established through path connectivity are invariant under topological transformations. When traversing the boundary of a disk, any set of points on the boundary forms a connected path. Under topological transformations, the sequence of the points along the boundary must be preserved. The boundary points of concern are the components of the boundary-boundary intersection. The *boundary-boundary sequence* describes the order in which the components of the boundary-boundary intersection occur.

Any discussion of a sequence (of components) requires a uniform *orientation*. Here, this is established through the fact that the plane, i.e., the embedding space $\mathbb{R}^2$, has been oriented and that A and B have orientations consistent with the orientation of the plane.

To define the boundary-boundary component sequence, $\partial A \cap \partial B$ is assumed to be non-empty and $\partial A \neq \partial B$. Otherwise, a discussion about sequences of the components of the boundary-

boundary intersections would be trivial as they have either none or exactly one, equal to the full boundary.

In the most general case of a non-empty boundary-boundary intersection, the sequence of the components is established by starting with an arbitrary boundary-boundary component, assigning it the number 0, and traversing ∂A in the orientation of the plane to the next boundary-boundary component, to which the next higher number is assigned, and continuing until the initial component, labeled number 0 is reached again. The same traverse in the orientation of the plane is performed along ∂B, this time recording the component numbers that were assigned when traversing ∂A. This process creates a sequence of (unique) numbers, called the *sequence of boundary-boundary components*. Figure 10 shows an example of an overlap relation between A_W and B_W in W whose boundary-boundary component sequence is <0; 2; 1; 3>.

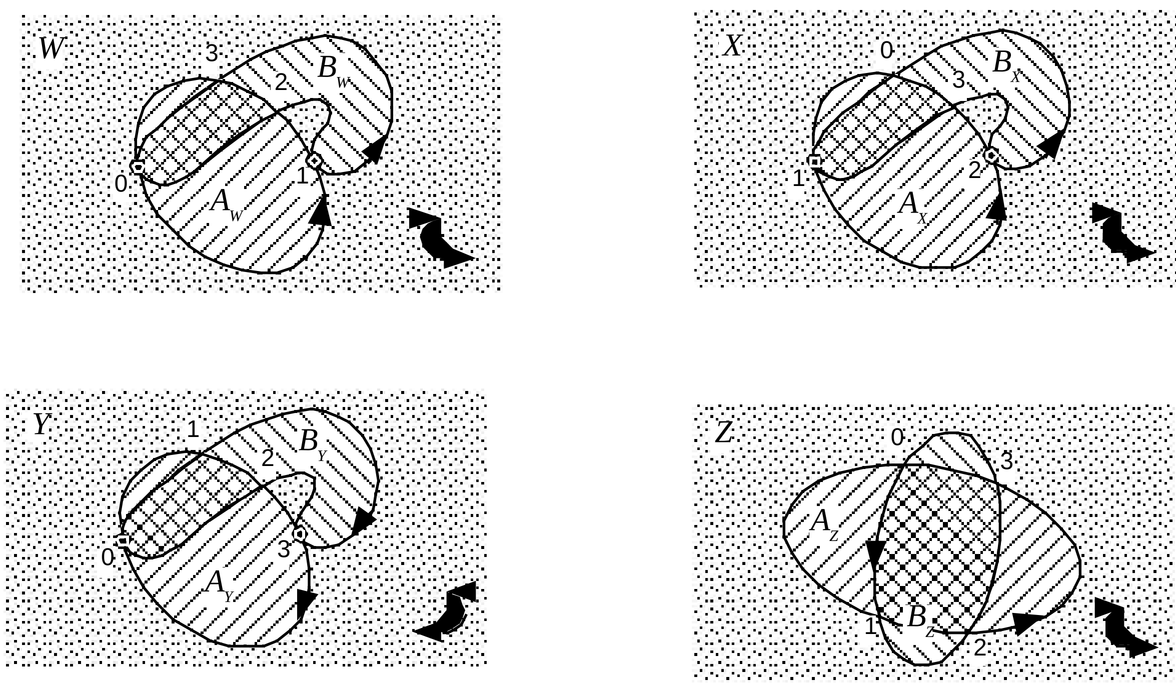

Figure 10: Configurations whose boundary-boundary component sequences are the same (W, X, and Y) and different (Z).

Obviously, the selection of the starting point and the orientation matters for the sequence, because different sequences may be obtained for the same configuration when starting the labeling of components at two different components or with two different orientations. The equivalence of two boundary-boundary sequences, S_1 and S_2, is therefore defined such that sequence S_1 must match with at least one of all possible sequences of S_2 obtained by its cyclic permutation and/or reversal. In a computational model, this can be achieved by applying $(n\text{-}i)$ mod n and/or iteratively applying $(i+1)$ mod n to each component number i, where n is the total number of components. For example, the boundary-boundary component sequence of the relation between A_X and B_X in X is <0; 1; 3; 2> (figure 10). Though the sequence is initially distinct from the sequence of the boundary-boundary components between A_W and B_W in W (<0; 2; 1; 3>) both are equivalent, because the labeling started only at a different component. Computationally, this can be found as follows:

(<0; 1; 3; 2> +*1*) mod *4* = <1; 2; 0; 3> ≡ <0; 3; 1; 2>
(<0; 3; 1; 2> +*1*) mod *4* = <1; 0; 2; 3> ≡ <0; 2; 3; 1>
(<0; 2; 3; 1> +*1*) mod *4* = <1; 3; 0; 2> ≡ <0; 2; 1; 3> (which is the exact sequence for *W* in figure 10).

Similarly, the sequence of the relation between A_X and B_X in X (figure 10) is <0,1,3,2>, and this can be seen to be equivalent to the one between A_W and B_W in W as follows:
(4 – <0; 1; 3; 2>) mod *4* = <0; 3; 1; 2>
(<0; 3; 1; 2> +*1*) mod *4* = <1; 0; 2; 3> ≡ <0; 2; 3; 1>
(<0; 2; 3; 1> +*1*) mod *4* = <1; 3; 0; 2> ≡ <0; 2; 1; 3> (which is the exact sequence for W in figure 10).

On the other hand, the boundary-boundary component sequence of between A_Z and B_Z in Z in figure 10, <0; 1; 2; 3>, is different from the sequences in *W*, *X*, and *Y*, because it cannot be transformed into <0; 2; 3; 1>, as all of its cyclic permutations result in the component sequence <0; 1; 2; 3>.

4.4 Types of boundary-boundary component intersections

Among the prototypes of the topological relations with non-empty boundary-boundary intersections (figure 2), one can find two important types of boundary-boundary intersections: (1) Boundary components whose connected arcs run through different parts of the other object. The relation *overlap* as depicted in figure 2 is an example of this configuration where both connected arcs of the boundary-boundary intersection are either the interior or exterior of the second object. (2) Boundary components whose connected arcs run through the same part of the other object—either the object's interior or exterior. This is the case for *meet*—the two arcs enter from and leave into the other object's exterior—and *covers* and *coveredBy* whose connected arcs run through the other object's exterior and interior, respectively. We define the two types of components as follows:

Definition 2: Let C be a component of the (non-empty) boundary-boundary intersection, $\partial A \cap \partial B$, in X. C is called a *crossing component* if for every neighborhood U of C, either both $\partial A \cap U \cap B^\circ$ and $\partial A \cap U \cap (X - B)$ are non-empty or both $\partial B \cap U \cap A^\circ$ and $\partial B \cap U \cap (X - A)$ are non-empty. A component that is not a crossing component is called a *touching component*.

Within the setting of 2-disks in IR^2, the crossing component types only arise for components of the boundary-boundary intersection. It is straightforward to prove that relations whose content invariant is (¬ , , ,), can have only boundary-boundary components of type touching. Likewise, (¬ , ¬ , ¬ ,) and (¬ , ¬ , , ¬) can have only touching boundary-boundary components. More challenging are configurations whose content invariant is (¬ , ¬ , ¬ , ¬), as they must have at least two boundary-boundary components of types crossing, but may have additional touching and crossing components. Figure 8 showed an example of such a topological relation in which two boundary-boundary components were of type crossing—C_0 and C_1—and another one was of type touching (C_2).

When considered in combination with other topological invariants, different types of the crossing classification may occur. The first refinement is due to the combination of the type of a boundary-boundary intersection and its dimension (see section 4.2). For 1-dimensional crossing components, there are two topologically distinct configurations possible, one in which the component intersects the interior of the union of the two disks (C_0 in figure 11), the other in which it lies completely in the boundary of the union (C_1 in figure 11); therefore, the two types of 1-dimensional crossings will be called respectively *inner crossing* and *outer crossing*.

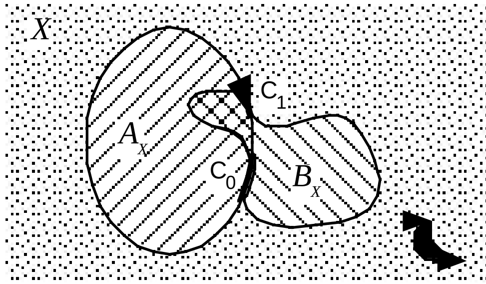

Figure 11: A configuration with inner crossing (C_0) and outer crossing (C_1) boundary-boundary components.

Another refinement of the crossing classification occurs when one considers the fixed orientation of the plane (which underlies the boundary-boundary component sequence, see section 4.3) and the type of the boundary-boundary intersection. The orientation implies an ordering of the boundary-boundary components and, when traversing *B*'s boundary, it gives rise to the distinction of boundary-boundary components that cross from *A*'s exterior *into A*'s interior (C_0 in figure 12), and those that cross *outOf A*'s interior (C_1 in figure 12). This invariant will be call the *crossing direction*.

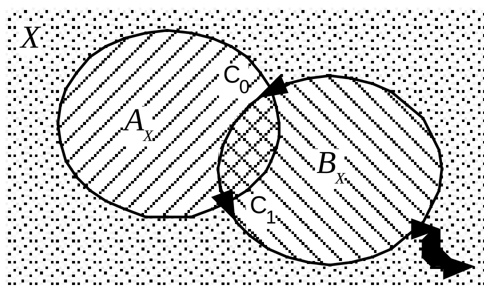

Figure 12: A configuration with boundary-boundary components that cross into *A* (C_0) and out of *A* (C_1) .

4.5 Complement relationship

Similar to the way the boundaries and interiors must preserve their topological properties among each other they have to retain topological properties with respect to the exterior (or complement). While it would be possible to analyze the relationships between each boundary and interior with the other object's exterior, adding another five intersections to the 4-intersection (Egenhofer and Herring 1991), we concentrate here on the relationships between the components of the boundary-boundary intersection and the exterior that is common to both objects. The *common exterior* is the complement of the union of two disks. It forms a single, connected open set for all those situations when the two disks are disjoint or equal and when *A* is inside *B*, *A* contains *B*, *A* covers *B*, or *A* is coveredBy *B*; however, in those situations when *A* and *B* overlap or *A* and *B* meet, the common exterior may have several components. These components are distinguished by the fact that exactly one of them is *unbounded*, while the others are *bounded*.

Those boundary-boundary components that intersect the boundary of the unbounded component of the common exterior are called the *unbounded components*, whereas the boundary-boundary components that do not intersect the unbounded exterior component will be referred to as the *bounded components*. The topological invariant that distinguishes between bounded and unbounded components will be called the *complement relationship*. Figure 13 shows an example of a meet relation in which C_0 and C_2 are unbounded components, while C_1 is an bounded component.

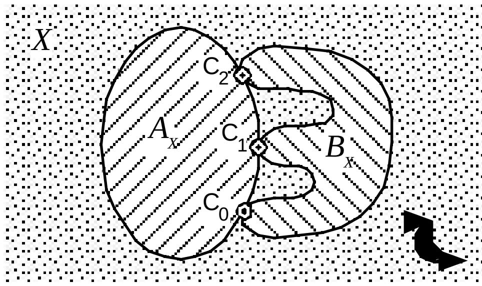

Figure 13: A configuration with two unbounded boundary-boundary components (C_0 and C_2.) and one bounded component (C_1).

5. Planar-disk classifying invariant

In this section, we will identify classifying invariants in terms of the component invariants. Classifying invariants are combinations of component invariants such that if two sets of 4-intersection invariants are the same for two pairs of 2-disks, then their topological relations are equivalent.

To evaluate a classifying invariant in the most general case, one records, while traversing ∂B, for each component a triple of the form:

component number (*component dimension*, *component type*, *complement relationship*)

where *component number* refers to the number assigned to the component as ∂A was traversed, *component dimension* is the dimension of the component (equal to either zero or one), *component type* indicates whether the component is touching or crossing—into A or outOf A, and, if necessary, inner crossing or outer crossing—and *complement relationship* describes whether the component is a bounded or unbounded component. This invariant is called the *planar-disk classifying invariant*.

For example, to compare the two topological relations between A_X and B_X and A_Y and B_Y in figure 14, one would construct the following two sequences of triples:

A_X and B_X: <0 (0, crossing outOf, unbounded); 4 (0, crossing into, unbounded);
5 (0, crossing outOf, unbounded); 6 (0, crossing into, unbounded);
3 (0, crossing outOf, bounded); 2 (1, touching, bounded);
1 (0, crossing into, bounded)>

A_Y and B_Y: <0 (0, crossing outOf, bounded); 4 (0, crossing into, bounded);
5 (0, crossing outOf, bounded); 6 (0, crossing into, bounded);
3 (0, crossing outOf, unbounded); 2 (1, touching, unbounded);
1 (0, crossing into, bounded)>

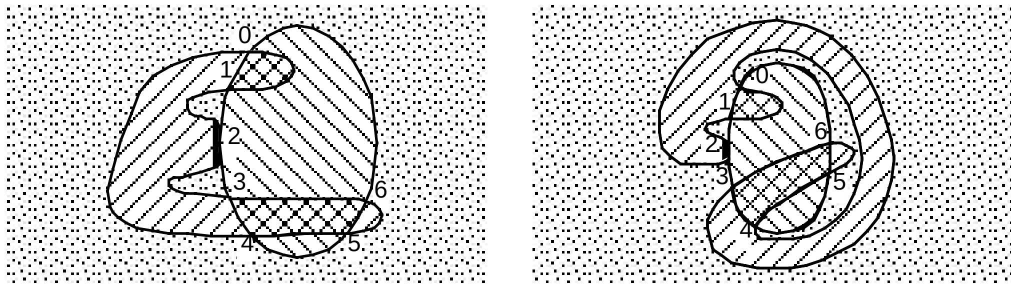

Figure 14: Two topologically distinct configurations that distinguish by different planar-disk invariants.

Theorem 1: The topological relations between disks *A* and *B* in the plane and between disks *A′* and *B′* in the plane are equivalent if and only if they have the same content invariant and, for non-empty boundary-boundary intersections, there is a choice of orientation of the plane and initial component of the boundary-boundary intersections such that the planar-disk classifying invariant for *A* and *B* is identical to the classifying invariant for *A′* and *B′*.

The proof of this theorem is based on the Schoenflies theorem (Thomassen 1992) and requires advanced topological results.

In the following we will prove that the four component invariants—sequence, dimension, type, and complement relationship—are independent, i.e., in general no combination of them would imply the value of another invariant. If one of the invariants were unnecessary then it could be replaced by a combination of the other three.

5.1 Need for sequence of boundary-boundary components

Recording only the types of the components, their dimensions, and their complement relationships leaves ambiguities about the actual topological relation, because it does not fix the sequence in which the triples of (*dimension, type, complement relationship*) may occur. Figure 15 shows an example of two configurations whose content invariants are both (¬ , ¬ , ¬ , ¬). Although there is a 1:1-correspondence between the triples of (*dimension, type, complement relationship*) for the boundary-boundary components—2 components with (0, crossing, unbounded), 2 components with (0, crossing, bounded), and 1 component with (0, touching, bounded)—the two relationships are topologically distinct, as can be seen by examining the number of interior-interior components: there are two interior-interior components between A_X and B_X in in X , whereas there are three between A_Y and B_Y in Y .

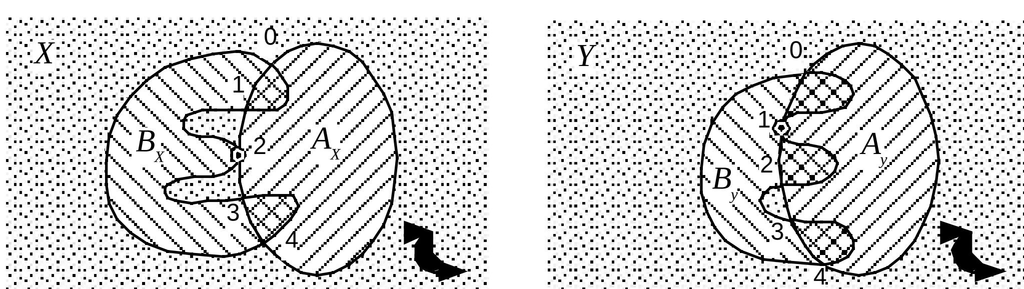

Figure 15: Two topologically distinct configurations that distinguish only by different boundary-boundary component sequences.

By including the sequence of the boundary-boundary components, one obtains enough information to distinguish between the two topological relations. The two sequences for figure 15 are:

<0 (0, crossing, unbounded); 4 (0, crossing, unbounded);
3 (0, crossing, bounded); 2 (0, touching, bounded);
1 (0, crossing, bounded)> (for *X* in figure 15)

<0 (0, crossing, unbounded); 4 (0, crossing, unbounded);
3 (0, crossing, bounded); 2 (0, crossing, bounded);
1 (0, touching, bounded)> (for *Y* in figure 15)

Even with a change in orientation or a change in the initial component of the boundary-boundary sequence, these invariants will always be different for the two configurations and, therefore, distinguish between these two topological relations.

5.2 *Need for component types*

The combination of sequence, dimension, and complement relationship also leaves enough freedom to have at least two different configurations. Let two relations have the same content invariant (¬ , ¬ , ¬ , ¬) and the sequences of their boundary-boundary components be both <0 (0, unbounded); 2 (0, bounded); 1 (1, unbounded)>. Figure 16 shows two configurations that have exactly these specifications, but are topologically distinct—the topological relation between A_X and B_X in X (figure 16) has one interior-interior component, whereas between A_Y and B_Y in Y there are two interior-interior components.

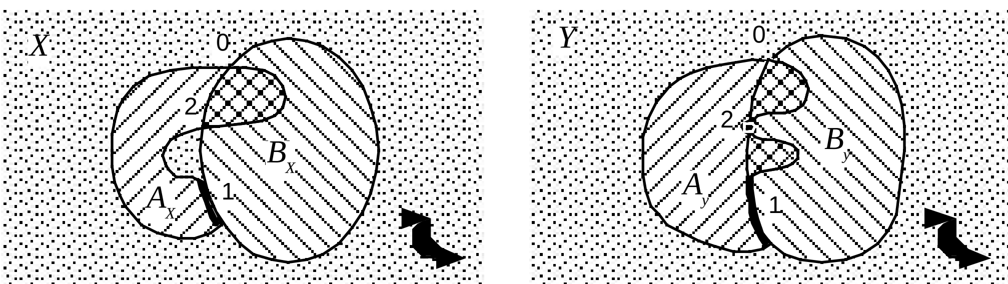

Figure 16: Two topologically distinct configurations that distinguish only by different component types (crossing vs. touching).

By adding the component type to the invariant, one can distinguish between the two configurations. For figure 16, these sequences of (*dimension, type, complement relationship*) are:

<0 (0, crossing, unbounded); 2 (0, crossing, bounded);
1 (1, touching, unbounded)> (for *X* in figure 16)
<0 (0, crossing, unbounded); 2 (0, touching, bounded);
1 (1, crossing, unbounded)> (for *Y* in figure 16)

5.2.1 Need for inner/outer crossing details: For relations with 1-dimensional crossing components, it is also necessary to distinguish whether the components are inner or outer crossings. For example, for an overlap relation with two components one could construct two topologically distinct configurations that have the same component invariants of *component number* (*dimension, type, complement relationship*), i.e., <0 (0, crossing, unbounded); 1 (1, crossing, unbounded)> (figure 17). This ambiguity can be overcome by recording for each 1-dimensional crossing component whether it is an inner or outer crossing:

<0 (0, crossing, unbounded); 1 (1, outer crossing, unbounded)> (for *X* in figure 17)
<0 (0, crossing, unbounded); 1 (1, inner crossing, unbounded)> (for *Y* in figure 17)

Figure 17: Two topologically distinct configurations that distinguish only by different inner/outer crossing types.

5.2.2 Need for into/outOf crossing details: For relations with crossing components, it is also necessary to distinguish whether the components cross into or outOf the reference disk. For example, the two configurations in figure 18 are topologically distinct since in *X*, A_X 's (directed) arc from component 3 to component 0 is located in the exterior of B_X , whereas in *Y*, the same arc is in B_Y 's interior; however, without specifying whether the components cross into or outOf, the resulting invariant sequences for the two configurations would be the same, i.e., <0 (0, crossing, unbounded); 1 (1, crossing, unbounded); 2 (1, crossing, unbounded); 3 (0, crossing, unbounded)>. By recording the crossing direction of the components one can resolve this ambiguity:

<0 (0, into crossing, unbounded); 1 (1, outOf crossing, unbounded);
2 (1, into crossing, unbounded); 3 (0, outOf crossing, unbounded)> (for *X* in figure 18)

<0 (0, outOf crossing, unbounded); 1 (1, into crossing, unbounded);
2 (1, outOf crossing, unbounded); 3 (0, into crossing, unbounded)> (for *Y* in figure 18)

Figure 18: Two topologically distinct configurations that distinguish only by different crossing directions.

5.3 Need for dimension

For each topological relation between two 2-disks, the combination of sequence, type, and complement relationship is sufficient to determine the number of components of the other intersections (i.e., interior-interior, interior-boundary, and boundary-interior); however, it does not distinguish completely between topological relations, because it disregards the difference between boundary-boundary components of dimension 0 and 1. For example, in figure 19 both topological relations have the same content invariant (¬ , ¬ , ¬ , ¬) and the same invariant of *component number* (*type*, *complement relationship*), i.e., <0 (crossing, unbounded); 1 (crossing, unbounded)>. Since one boundary-boundary component in X between A_X and B_X has dimension 1, while none in Y between A_Y and B_Y does, it is clear that the topological relations are distinct. By adding the dimension of each boundary-boundary component to the invariant, one can distinguish between these topological relations:

<0 (1, crossing, unbounded); 1 (0, crossing, unbounded)> (for X in figure 19)
<0 (0, crossing, unbounded); 1 (0, crossing, unbounded)> (for Y in figure 19)

Figure 19: Two topologically distinct configurations that distinguish only by different component dimensions.

5.4 Need for complement relationship invariant

Finally, it will be shown that the complement relationship is independent of the dimension, type, and sequence of components and, therefore, not implied by their combination. Consider the two configurations in figure 20. They are topologically distinct because, for instance, in X between A_X and B_X, the arc from component 0 to 2 is part of B_X's boundary separating B_X's interior from the unbounded exterior, whereas in Y between A_Y and B_Y, the same arc separates B_Y's interior from the bounded exterior. The invariant of *component number* (*dimension*, *type),* however, yields the same result in both cases, i.e., <0 (1, touching); 2 (0, crossing); 3 (0, crossing); 4 (0, crossing); 1 (0, crossing)>. By adding the values for the complement relationship, one obtains an invariant that distinguishes between the two relations:

<0 (1, touching, unbounded); 2 (0, crossing, unbounded);
3 (0, crossing, unbounded); 4 (0, crossing, unbounded);
1 (0, crossing, bounded)> (for X in figure 20)

<0 (1, touching, unbounded); 2 (0, crossing, bounded);
3 (0, crossing, bounded); 4 (0, crossing, bounded);
1 (0, crossing, unbounded)> (for Y in figure 20)

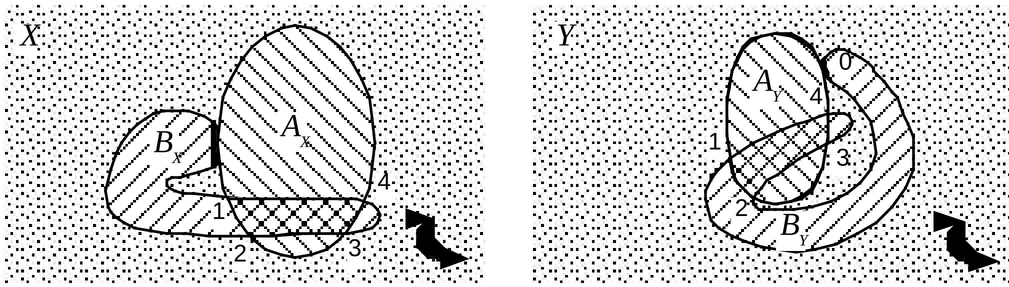

Figure 20: Two topologically distinct configurations that distinguish only by different complement relationships.

6. Dependencies between content and planar-disk classifying invariant

While in the most general case the entire set of all four component invariants is necessary, there are situations in which the values of some of the component invariants are implied by other component invariants or combinations thereof. Most obvious are the situations in which the content invariant of the boundary-boundary intersection is empty, as no further refinements are possible for an empty set. Likewise, there are no further refinements for the topological relation equal. The other relations with non-empty boundary-boundary intersections have the following consistency constraints for their component invariants:

- If *A* and *B* meet, only the sequence of bounded and unbounded components and their dimensions provide relevant information. The type of components is implied as all boundary-boundary components must be touching components. Even the order of the boundary-boundary components is uniquely determined: numbered 0,…, *n* along the boundary of *A*, the sequence of components when traversing *B* in the same orientation must be 0, *n*, *n*-1,…, 1.

- If *A* covers *B*, only the sequence of the dimension of the components matters. All boundary-boundary components must be touching and unbounded components. Also, the sequence of the boundary-boundary components, established along *A*'s boundary as 0,…, *n* implies that the sequence along *B*'s boundary, traversed in the same orientation, is 0,…, *n* as well. The same holds for all configurations in which *A* is coveredBy *B*.

Table 1 summarizes which component invariants apply to the boundary-boundary components for different configurations classified by the content invariant.

	dimension	sequence	complement relationship	type	boundedness	crossing direction
equal						
disjoint						
inside						
contains						
covers	×	×				
coveredBy	×	×				
meet	×	×	×			
overlap	×	×	×	×	×†	×†§

† Only for crossing boundary-boundary components.

§ Only for 1-dimensional boundary-boundary components.

Table 1: Component invariants that allow to make finer distinctions of relations realized with the content invariant.

7. Conclusions

We presented a formalism to determine a classifying invariant for 2-disks in the plane. The formalism is an extension of the 4-intersection invariant with empty/non-empty intersections. While the content is sufficient to distinguish uniquely among any topological relation with empty boundary-boundary intersections, it is necessary to consider additional 4-intersection invariants for most non-empty boundary-boundary intersections. The 4-intersection invariants considered here were invariants of the components of boundary-boundary intersections: the types, dimensions, relationships with respect to the complement, and their sequences. It was shown that in the most general case of "overlap" relations, i.e., relations whose content invariant is (¬ , ¬ , ¬ , ¬), all four component invariants are necessary to infer from two equal sets of invariants that the corresponding topological relations are identical.

The results presented may serve as the most detailed level of a topological data model for disk-like objects in the plane. With the 4-intersection invariants of content, boundary-boundary components, their dimensions, types, complement relationships, and sequences it is possible to identify any two equivalent topological relations between two such objects. This is relevant for spatial query processing as any query about topological relations between two disks can be derived from it. On the other hand, the model allows for generalizations, depending on the details desired. For example, one may describe a certain level of similarity of topological relations by fixing the number of components and their types, but disregarding their dimension. Likewise, one may define different classes of similar/dissimilar relations by forming groups, such as overlap-A: "exactly two boundary-boundary components," overlap-B: "between two and four crossing boundary components," and overlap-C: "more than four crossing boundary components."

The results of this paper open new avenues for the investigation of a series of interesting research questions:

Which invariants are implied by considering the components of boundary-boundary intersection and interior-interior intersection?

The model presented here and the invariants chosen were guided by mathematical principles. From a cognitive perspective, it would make sense to analyze not only the boundary-boundary components, but also the components of the interior-interior intersection, and—if necessary—the components of the interior-boundary and boundary-interior intersections. The consideration of the other components may imply that some invariants, necessary when using the boundary-boundary components only, are implied by the other components. Such alternative models would be of great benefit when comparing complexity and computational cost for different implementation strategies.

What are the priorities among the different invariants?

All invariants discussed in this paper are refinements that apply only if certain intersections were non-empty. The discussion of the dimension as an invariant of the boundary-boundary components also showed that it adds little to the stability of a relationship, i.e., leaving away the dimension as an evaluation criterion may cause only minor topological differences. So content and dimension could be rated as the strongest and weakest invariant, respectively. Are there any priorities among the other invariants?

What are the basic component invariants necessary to establish a classifying invariant for topological relations between lines in IR^2 *or between a region and a line in* IR^2*?*

It has been shown that in order to distinguish such binary topological relations, it is necessary to consider the nine intersections of the two interiors, boundaries, and exteriors (Egenhofer and Herring 1991); in addition, such topological invariants as the components, their dimensions, types, and sequences may be necessary to describe equivalence. As pointed out by Herring (1991), the type of intersections may be a crucial invariant when analyzing relationships between objects embedded in higher spaces such as two (1-dimensional) lines in IR^2. For such configurations, however, the component type would apply to intersections with the interiors, rather than to components of the boundary-boundary intersection. Likewise, for topological relations between a region and a line (Mark and Egenhofer 1992), the type of the component intersection applies to the intersections between the interior of the lower-dimensional object (i.e., the line) and the boundary of the higher-dimensional object (i.e., the region).

How do the detailed invariants influence the composition and consistency of topological relations?

The additional details about the meet, covers, coveredBy, and overlap relations will influence the composition of topological relations. The composition table for region relations, based exclusively on the content invariant (Egenhofer 1991), has a number of underdetermined compositions, which may be reduced if detailed topological information is available and, therefore, lead to more precise topological inferences.

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