International Journal of Geographical Information Systems 8 (2): 129-144, 1994.

Topological relations between regions with holes*

MAX J. EGENHOFER†, ELISEO CLEMENTINI, and PAOLINO DI FELICE

Università di L'Aquila, Dipartimento di Ingegneria Elettrica, 67040 Poggio di Roio, L'Aquila, Italy

Abstract

The 4-intersection, a model for the representation of topological relations between 2-dimensional objects with connected boundaries and connected interiors, is extended to cover topological relations between 2-dimensional objects with arbitrary holes, called *regions with holes*. Each region with holes is represented by its generalized region—the union of the object and its holes—and the closure of each hole. The topological relation between two regions with holes, A and B, is described by the set of all individual topological relations between (1) A's generalized region and B's generalized region, (2) A's generalized region and each of B's holes, (3) B's generalized region with each of A's holes, and (4) each of A's holes with each of B's holes. As a side product, the same formalism applies to the description of topological relations between *1-spheres*. An algorithm is developed that minimizes the number of individual topological relations necessary to describe a configuration completely. This model of representing complex topological relations is suitable for a multi-level treatment of topological relations, at the least detailed level of which the relation between the generalized regions prevails. It is shown how this model applies to the assessment of consistency in multiple representations when, at a coarser level of less detail, regions are generalized by dropping holes.

1. Introduction

The concepts and notions of *topological relations* have been discussed in the GIS literature for a number of years. There are now several different models available that deal with spatial relations among such simple objects as homogeneously 2-dimensional, connected areas (Egenhofer and Franzosa 1991; Herring 1991; Randell *et al.* 1992; Clementini *et al.* 1993) and lines with exactly two endpoints (Egenhofer, in press). Numerous prototypes and implementations in commercial systems have been reported (Herring 1991; de Hoop and van Oosterom 1992; Keighan 1993). While these models of topological relations have become very popular, they are based on a simplified spatial data model. In general the objects treated are simple since they have such constraints as "there must be a single connected boundary for each areal object," "each object must be connected," and "each line must have exactly 2 endpoints." In order to become operational within a GIS environment, it is necessary that the existing theories cover the kinds of geometric objects that are necessary for geographic applications, not just a simplified subset thereof.

GISs deal with a model of reality and describe such geographic objects and spatial relations as, "San Marino is surrounded by Italy," "Italy is a neighbor of Switzerland," and "Greece is a part of the European Community." The geometry of these objects is frequently irregular since it has

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* This work was partially supported by the Italian National Council of Research (CNR) under grant No. 92.01574.PF69. Max Egenhofer's work is partially supported by Intergraph Corporation, NSF grant No. IRI-9309230, and the NCGIA under NSF grant No. SES-8810917.

† On a leave of absence from the National Center for Geographic Information and Analysis, Department of Surveying Engineering, and Department of Computer Science, Boardman Hall, University of Maine, Orono, ME 04469-5711, USA.

"exceptions" of a more simplistic treatment of geometry such as holes and separations. Such constraints have been treated extensively at the level of representing spatial objects (Frank and Kuhn 1986; Herring 1987; Egenhofer *et al.* 1989; Güting and Schneider 1993), but have been often excluded in the treatment of spatial relations. Some investigations of the relations between more complexly structured geographic objects exist such as topological relations with lines that have more than two endpoints (Egenhofer and Herring 1991), and topological relations that consider a generalization of the shape of an areal object to its convex hull (Cui *et al.* 1993). Work that comes closest to these investigations describes some topological relations between regions with holes (Egenhofer and Franzosa 1991) and develops a hierarchical model for areal objects with holes (Worboys and Bofakos 1993).

This paper contributes to the treatment of spatial relations among complexly structured spatial objects. Its scope is the formalization of topological relations between 2-dimensional objects that may contain *holes*. While the objects of concern may have disconnected boundaries and disconnected exteriors, their closures must be generally connected. Objects with disconnected closures would form separations, which are excluded from these investigations. We base our work on previous results of the formalization of topological relations between regions (homogeneously 2-dimensional objects with connected boundaries and connected interiors) (Egenhofer and Franzosa 1991) and the assessment of topological consistency of a scene description (Egenhofer and Sharma 1993).

These investigations are part of a larger research effort, the assessment of *consistency in multiple representations*. Multiple representations encompass changes in the geometric and topological structure of a digital object that may occur with the changing resolution at which that object is encoded for computer storage, analysis, and depiction (Buttenfield 1989). The process that derives the more general representation from a more detailed one is frequently called *generalization* (Bruegger and Frank 1989; Buttenfield and McMaster 1991). Current GISs are based on single-representation models, however, there are many GIS applications that have to deal with multi-resolution data. For example, vehicle navigation systems require significantly different representations of the same data for planning a trip, giving instruction, and driving (Timpf *et al.* 1992). A major impediment in the transition to more powerful multiple-representation GISs is the lack of methods to maintain consistently the multiple representations of geographic objects. We are particularly concerned with the consistent modeling of *spatial relations among several objects* where each object is represented at multiple levels of details.

Within this setting of multiple representations, an important aspect is the strategy to generalize complexly structured spatial objects by dropping their holes. When dropping one or several holes of an object, it is important that the object's topological relations with respect to all other objects be maintained. Any such inconsistency may cause serious problems when queries get processed against one or the other representation. The results of this paper provide a means to assess whether or not such a generalization maintains the major topological properties among the objects involved.

The remainder of this paper is structured as follows: Section 2 gives a brief summary of the model of topological relations between regions and demonstrates why it is insufficient as a model for topological relations between regions with holes. Section 3 defines the concept of a region with holes that must be fully surrounded by some interior, and introduces a model for the topological relations between such objects. Section 4 shows how the topological relations among such regions with embedded holes can be expressed by the description of a scene of topological relations among multiple regions, and section 5 derives the smallest amount of information that is necessary to describe completely the topological relations between two regions with embedded holes. Section 6 relaxes the definition of a region with holes to include holes along the region's boundary and holes that touch each other. Section 7 analyzes how this model applies to the assessment of topological consistency among multiple representations. The conclusions in Section 8 summarize the major results.

2. Topological relations

Topological relations are spatial relations that are preserved under such transformations as rotation, scaling, and rubber sheeting. The model for binary topological relations used in this paper is based on the usual concepts of point-set topology with open and closed sets (Alexandroff 1961). A *region* is a homogeneously 2-dimensional point set embedded in IR^2 with a connected interior, denoted by A°, a connected boundary, denoted by ∂A, and a single connected exterior, denoted by A^-. The definition of binary topological relations between two such regions, A and B, is based on the four intersections of A's boundary and interior with the boundary and interior of B (Franzosa and Egenhofer 1992). A 2×2 matrix, called the 4-intersection, concisely represents these criteria (Equation 1).

$$\begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^\circ \\ A^\circ \cap \partial B & A^\circ \cap B^\circ \end{pmatrix} \tag{1}$$

By considering the values empty (ø) and non-empty (¬ ø), one can distinguish between sixteen binary topological relations, eight of which can be realized for two regions with connected boundaries if the objects are embedded in IR^2 (Egenhofer and Herring 1990). They are called *disjoint*, *meet*, *equal*, *inside*, *contains*, *covers*, *coveredBy*, and *overlap* (figure 1). This set provides a complete coverage and is mutually exclusive so that exactly one of these topological relations holds true between any two regions (Egenhofer and Franzosa 1991). For each topological relation r_i there exists a *converse* relation r_j such that $r_i(A, B) = \breve{r}_j(B,A)$.

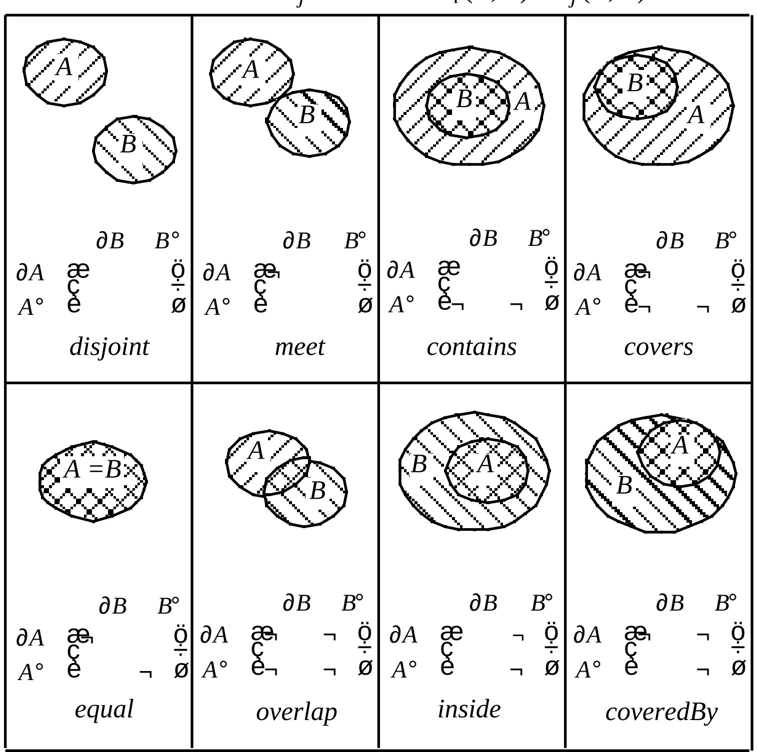
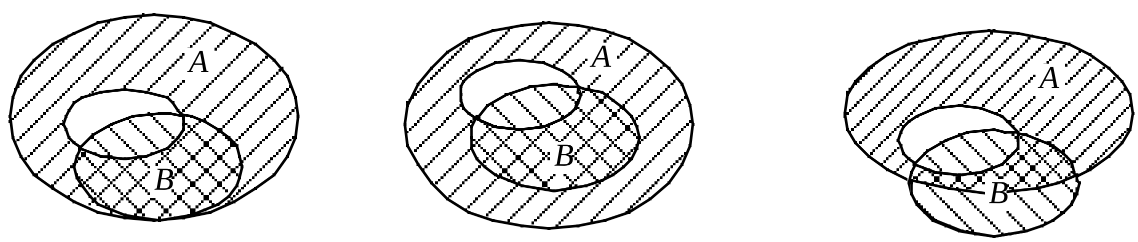


Figure 1: The eight topological relations between two regions with connected boundaries for the 4-intersection.

Subsequently, the disjunction of all eight topological relations will be referred to as the *universal relation*, denoted by $\boldsymbol{U}$. The *composition* of topological relations forms the basis for an algebra to reason about topological relations (Egenhofer 1991; Smith and Park 1992). For example, if region A *meets* region B, and B *contains* region C, then A and C must be *disjoint*. The composition of two relations r_i and r_j will be denoted by $r_i ; r_j$, e.g.,

$$meet ; contains = disjoint \tag{2}$$

The 4-intersection is used extensively, e.g., for studies that describe spatial relations between other kinds of objects such as lines and volumes (Egenhofer and Herring 1991; Herring 1991; Pigot 1991; Hadzilacos and Tryfona 1992; Hazelton *et al.* 1992; Randell *et al.* 1992; Clementini *et al.* 1993), in spatial query languages (Svensson and Zhexue 1991; de Hoop and van Oosterom 1992), to deduce spatial information (Smith and Park 1992; Abdelmoty *et al.* 1993), to assess topological consistency (Egenhofer and Sharma 1992), and as a basis for cognitive-linguistic studies (Mark and Egenhofer 1992). While the 4-intersection treats sufficiently the topological relations between two regions, it provides only a very limited service to distinguish between topological relations with objects that have holes. For example, figure 2 shows three configurations of topological relations between 2-dimensional objects with holes, all of which have the same 4-intersection, but have topologically distinct relations. Even by considering the intersections with the exteriors (Egenhofer and Herring 1991) one cannot distinguish between the three configurations.



(a) (b) (c)

Figure 2: Three topologically distinct relations between 2-dimensional objects with holes, all of which have the 4-intersection $\begin{matrix} & \partial B & B^\circ \\ \partial A & \neg\emptyset & \neg\emptyset \\ A^\circ & \neg\emptyset & \neg\emptyset \end{matrix}$.

3. Definition of regions with embedded holes

Unlike a region without holes, the exterior of a region with holes may be separated. Separations of the exterior imply that there exists one *outer exterior* (unbounded set) and *n>0 inner exteriors* (bounded sets). The outer exterior will be denoted by A_0^- and the inner exteriors by $A_1^- \ldots A_n^-$. Their union makes up the entire exterior, i.e.,

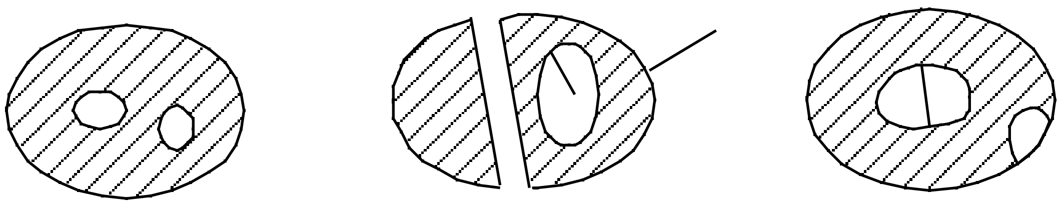
$$A^- = \bigcup_{i=0}^{n} A_i^- \tag{3}$$

Definition 1: A *region with embedded holes*, denoted by A, is a non-empty subset of IR^2 with a connected interior such that the closure of any two different exteriors are disjoint (Equation 4) and A is equal to the closure of A's interior (Equation 5).

$$i, j = 0 \quad n, i \neq j: \quad \overline{A_i^-} \cap \overline{A_j^-} = \tag{4}$$

$$A = \overline{A^\circ} \tag{5}$$

This definition includes regions with holes that are completely surrounded by the interior of the region (figure 3a); however, it excludes spikes and separations of the interior (figure 3b). Likewise, it excludes holes that touch the boundary or another hole (figure 3c).



(a) (b) (c)

Figure 3: A possible (a) and two impossible (b and c) regions with embedded holes.

Definition 2: A *hole* of A, denoted by H^A, corresponds to the closure of an inner exterior, i.e., $H_i^A = \overline{A_i^-}$ for $i > 0$.

A hole is a connected set that is strictly contained in A and each hole H_i^A is disjoint from all other holes H_j^A in A with $i \neq j$.

Definition 3: Considering n holes $H_1^A \ldots H_n^A$ in A, the *generalized region* A^* is defined as the union of A and all holes contained in A, i.e.,

$$A^* = A \quad \left(\bigcup_{i=1}^{n} H_i^A \right) \tag{6}$$

This implies for a region without holes that $A^* = A$.

The concept of a hole as the closure of an inner exterior allows us to map any region with embedded holes A into a group of simple regions A^*, H_1^A, …, H_n^A, each without any holes. By considering the holes as separate objects, the modeling of topological relations between regions with embedded holes can be expressed in terms of topological relations between simple regions. Therefore, the problem of modeling region-with-hole relations corresponds to modeling the relations in a scene of regions (Egenhofer and Sharma 1993). The topology of a scene of n objects is fully specified by n^2 topological relations. They can be concisely represented by a $n \times n$ matrix M, called the *relation matrix*.

Let A and B be two regions with *m* and *n* holes, respectively. The objects of the scene description for these two regions with embedded holes consists of the set $S = \{A^*, H_1^A, \quad , H_m^A, B^*, H_1^B, \quad , H_n^B\}$ with (1) the generalized region of A, (2) the generalized region of B, (3) each hole in A, denoted by $H_1^A, \ldots, H_m^A$, and (4) each hole in B, $H_1^B, \ldots, H_n^B$. Since the scene description is made up of $(m+n+2)$ simple regions, the total number μ of topological relations that can be specified between the objects and their holes is:

$$\mu = (m + n + 2)^2 \qquad (7)$$

Figure 4 shows an example configuration of two regions with one and three holes, respectively and table 1 gives the corresponding relation matrix *M*, which denotes all binary topological relations among the $(m+n+2)$ objects.

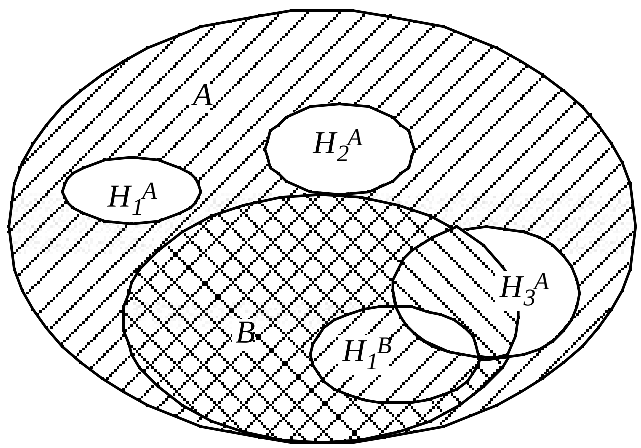


Figure 4: Example of the topological relation between two objects with embedded holes, A with holes H_1^A, H_2^A, and H_3^A; and B with H_1^B.

In this notation, an individual relation will be referred to as *M* [a, b], e.g., *M* [H_1^A, A^*] = *inside.*

	A^*	H_1^A	H_2^A	H_3^A	B^*	H_1^B
A^*	equal	contains	contains	contains	covers	contains
H_1^A	inside	equal	disjoint	disjoint	disjoint	disjoint
H_2^A	inside	disjoint	equal	disjoint	meet	disjoint
H_3^A	inside	disjoint	disjoint	equal	overlap	overlap
B^*	coveredBy	disjoint	meet	overlap	equal	contains
H_1^B	inside	disjoint	disjoint	overlap	inside	equal

Table 1: The relation matrix for the scene in figure 4.

4. Eliminating redundant relations

As for any scene representation, the entire set of $(m+n+2)^2$ topological relations is redundant (Egenhofer and Sharma 1993). The relations that can be immediately eliminated are those that are enforced by the node consistency and the arc consistency (Mackworth 1977).

- The relation between each object and itself must be *equal* (node consistency).
- The relation between A and B must be equal to the converse relation between B and A (arc consistency).

These two constraints reduce the number of relations required to fully describe the scene to μ' (Equation 8).

$$\mu' = \frac{(m+n+2)^2 - (m+n+2)}{2} \tag{8}$$

Furthermore, there are certain constraints about the topological relations that must hold between each generalized region and its holes, and among the holes of the same region.

- The topological relation between the generalized region and each of its holes must be always *contains*.
- The topological relation between any pair of holes that belong to the same object must be *disjoint*.

Therefore, the number of relations can be further reduced to μ'' (Equation 9).

$$\mu'' = mn + m + n + 1 \tag{9}$$

This leaves only those relations between A^* and B^*; A^* and B's holes; A's holes and B^*; and A's holes and B's holes. It applies to any configuration independent of the particular values the topological relations may have. For example, in order to describe the topological relation between the region A with three holes and region B with one hole (figure 4), it is sufficient to specify eight topological relations. Table 2 shows the reduced relation matrix for this scene, in which all implied relations have been left out.

	A^*	H_1^A	H_2^A	H_3^A	B^*	H_1^B
A^*					covers	contains
H_1^A					disjoint	disjoint
H_2^A					meet	disjoint
H_3^A					overlap	overlap
B^*						
H_1^B						

Table 2: The reduced relation matrix for the scene in figure 4.

Subsequently, only the part with the necessary relations will be displayed, called the *explicit relations* among the generalized regions and their holes. From the notation like H_1^A and H_2^A one can infer the remaining relations—the *implicit relations* among the generalized regions and their holes—as follows:

- H_1^A *inside* A^*,
- H_2^A *inside* A^*, and
- H_1^A *disjoint* H_2^A.

5. Minimizing the number of necessary relations

In the most general case, the explicit relations represent the smallest set of relations that are necessary to describe a binary topological relation between regions with embedded holes; however, depending on the configuration and the particular values of the relations, some of the remaining topological relations may be inferred as well. For example, for the configuration in figure 4, the fact that hole H_1^A is *disjoint* from B^* implies that hole H_1^A must be also *disjoint* from hole H_1^B.

Egenhofer and Sharma (1993) presented an algorithm to determine whether an incompletely observed scene description of topological relations can be completed by representing the set of all topological relations as a network. It iterates over the relation matrix M and applies the path-consistency constraint—the intersection of all possible compositions of path-length 2—until no additional inferences can be made. The completeness of a scene is finally evaluated by testing whether each inferred relation is unique, i.e., its cardinality ($\#$) is equal to 1 (Algorithm 1).

Algorithm 1: sceneIsComplete (M: relationMatrix): boolean
$M' := M$
REPEAT $M := M'$
$$ij: \quad M'[i,j] := \bigcap_{k=0}^{n} (M[i,k]\,;\, M[k,j])$$
UNTIL $M = M'$
sceneIsComplete := $ij: \quad \#(M'[i,j]) = 1$
end sceneIsComplete

Here, the reverse process will be applied, reducing the set of relations to its smallest number so that the entire scene description can still be inferred from the combination of the remaining topological relations. Such a set will be referred to as a *smallest set* of base relations that describe the topological relation between two regions with embedded holes. Due to the properties of the composition table of topological relations (Egenhofer 1991), a scene description has a single minimal set if and only if the elements in M's diagonal are *equal*. To find such a set , we start with the set of the implicit and explicit relations (the relation matrix M), which is assumed to be node- and arc-consistent. The following iterative algorithm reduces the set of relations and finds a smallest set of relations, from which all remaining topological relations between two regions with embedded holes can be inferred (Algorithm 2). Since the goal is to reduce the number of explicit relations, the minimization iterates only over these relations. Iteratively, each explicit relation and its converse are set to "unknown" (i.e., the universal relation $\boldsymbol{U}$) and it is tested whether this reduced set of relations would be sufficient to determine the scene completely. If so, then it is tried to eliminate recursively additional relations. Otherwise, the value for the explicit relation is kept and it is tried to eliminate the next. The processing stops when all explicit relations have been visited. A smallest set of explicit relations is the set with the greatest number of unknown relations. It must have been initialized with the completely observed relation matrix and gets updated whenever a smaller set has been found.

```
Algorithm 2:  minimize (M: relationMatrix, smallestSet: relationMatrix)
              FOR i := A* TO H_m^A DO
                  FOR j := B* TO H_n^B DO
                      IF M[i, j] ≠ U THEN
                          M' := M
                          M'[i, j] := U
                          M'[j, i] := U
                          IF sceneIsComplete (M') THEN
                              IF #(M) < #(smallestSet) THEN smallestSet := M'
                              minimize (M', smallestSet)
              end minimize
```

The benefits from the additional reduction of necessary explicit relations vary and depending on the particular configuration, either none or almost all explicit relations may be found to be redundant. An example of a configuration in which no further reduction are possible is when all explicit relations are *overlap*. On the other hand, there are configurations in which a smallest set of necessary relations can be reduced to a single topological relation (independent of the number of holes). For example, for the configuration in figure 5 it is sufficient to record a single topological relation, the one between H_1^A and B^*, because all other implicit and explicit relations between the generalized regions and the holes can be inferred.

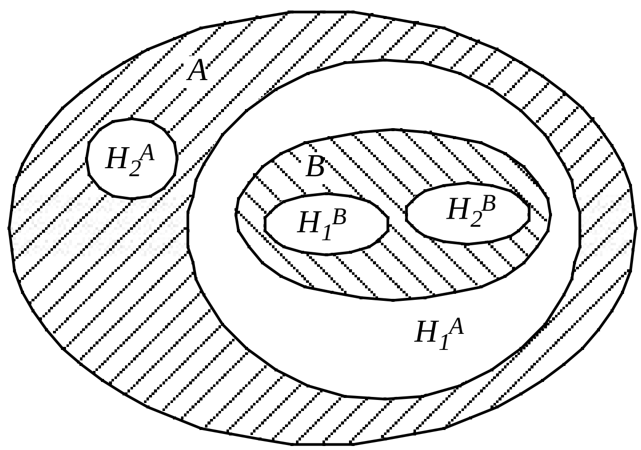


Figure 5: A configuration for which it is sufficient to record explicitly the topological relation between H_1^A and H_1^B.

6. Relaxing the constraints about holes

Initially, the constraints among the generalized region and its holes were defined such that each hole had to be strictly contained in the region. Likewise, any pair of holes of the same region had to be disjoint. In this section, these constraints will be relaxed, allowing the holes to be at the region's boundary or along the boundary of another hole. This relaxation requires a modification of the definition for the *region with embedded holes* (Definition 1), while the definitions of the *hole* (Definition 2) and *generalized region* (Definition 3) remain the same.

Definition 4: A *region with holes*, denoted by A, is a non-empty subset of $\mathbb{R}^2$ with a connected interior such that the union of the region and all holes is equal to the closure of the union of the region's interior and all its inner exteriors (10), and the interior of any hole is equal to the corresponding inner exterior (11).

$$A \cup \left(\bigcup_{i=1}^{n} H_i^A\right) = \overline{A^\circ \cup \left(\bigcup_{i=1}^{n} A_i^-\right)} \qquad (10)$$

$$\forall i = 1 \ldots n: \ \left(H_i^A\right)^\circ = A_i^- \qquad (11)$$

Equation 10 is weaker than equations 4 and 5 together, since it allows for holes like those shown in figure 3c that were disallowed by definition 1. While Equation 10 excludes spikes in the exterior, Equation 11 is necessary to exclude any spikes in the holes.

This extension influences the discussion about eliminating redundant relations (Section 4), since it implies a change in the types of topological relations that may hold between a generalized region and its holes, and among the holes of the same region:

- The topological relation between the generalized region and each of its holes is *contains covers equal*.
- The topological relation between any pair of holes that belong to the same object is *disjoint meet*.

Therefore, the reduced relation matrix has to consider also the topological relations above. For example, in order to describe the topological relation between the region A with three holes and region B with one hole (figure 6), it is necessary to specify the upper triangular relation matrix shown in table 3.

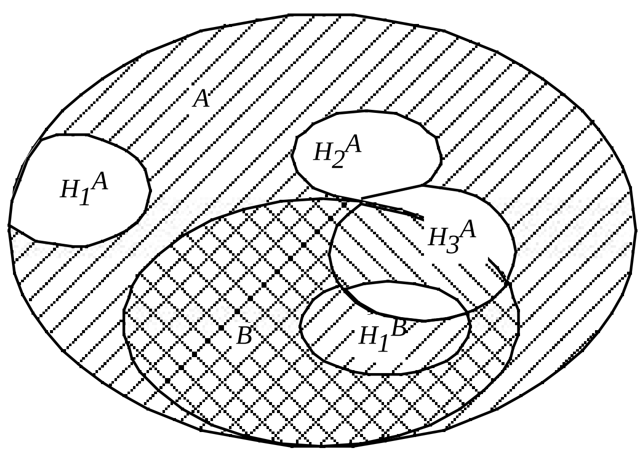
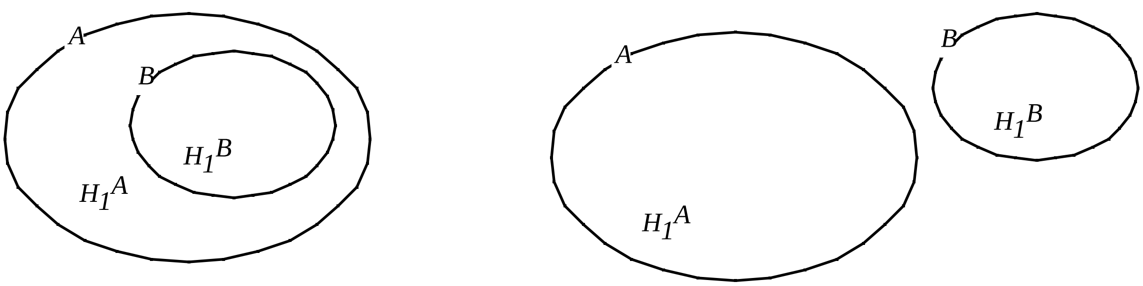


Figure 6: A configuration with holes along the boundary and holes that touch each other.

	A^*	H_1^A	H_2^A	H_3^A	B^*	H_1^B
A^*		covers	contains	contains	covers	contains
H_1^A			disjoint	disjoint	disjoint	disjoint
H_2^A				meet	meet	disjoint
H_3^A					overlap	overlap
B^*						contains
H_1^B						

Table 3: The relation matrix for the configuration in figure 6.

As a side product, this formalization of regions with holes provides also a comprehensive solution for the treatment of *1-spheres* and their topological relations. A 1-sphere is the boundary of a 2-disk (Spanier 1966). As such, it is a line with an empty boundary and an interior that coincides with its closure. In the present model, a 1-sphere S would be represented as follows: $S°$ maps onto the boundary of a region A, which in turn coincides with a single hole H_1^A such that $A° =$ and $\partial A = A$; therefore, the closure of A's inner exterior is equal to A^*. In the relation matrix, the value between A^* and H_1^A is *equal*, and therefore, the topological relation between any other region and A^* determines the topological relation with H_1^A. Figure 7 depicts two topologically distinct relations between two pairs of spheres, and Table 4 shows their respective relation matrices.



(a) (b)

Figure 7: Two topological relations between two spheres.

	A^*	H_1^A	B^*	H_1^B
A^*		equal	contains	contains
H_1^A			contains	contains
B^*				equal
H_2^A				

(a)

	A^*	H_1^A	B^*	H_1^B
A^*		equal	disjoint	disjoint
H_1^A			disjoint	disjoint
B^*				equal
H_2^A				

(b)

Table 4: The relation matrices for the configuration between the spheres displayed in figure 7.

7. Application to multiple representations

This extension of modeling topological relations is important for the consistent treatment of relations among objects that are represented at multiple levels of detail. A common strategy used in multiple representations is that holes are dropped when progressively moving through more general levels of detail. As such, a change in the number of holes through consecutive multiple representation levels must be monotonically decreasing and at the most generalized level of detail, regions are represented without any holes.

Figure 8 shows the sequences of all possible generalization steps that eliminate holes, starting at the most detailed level with regions A with two holes and B with one hole, and finishing at the least detailed level where both regions have no holes.

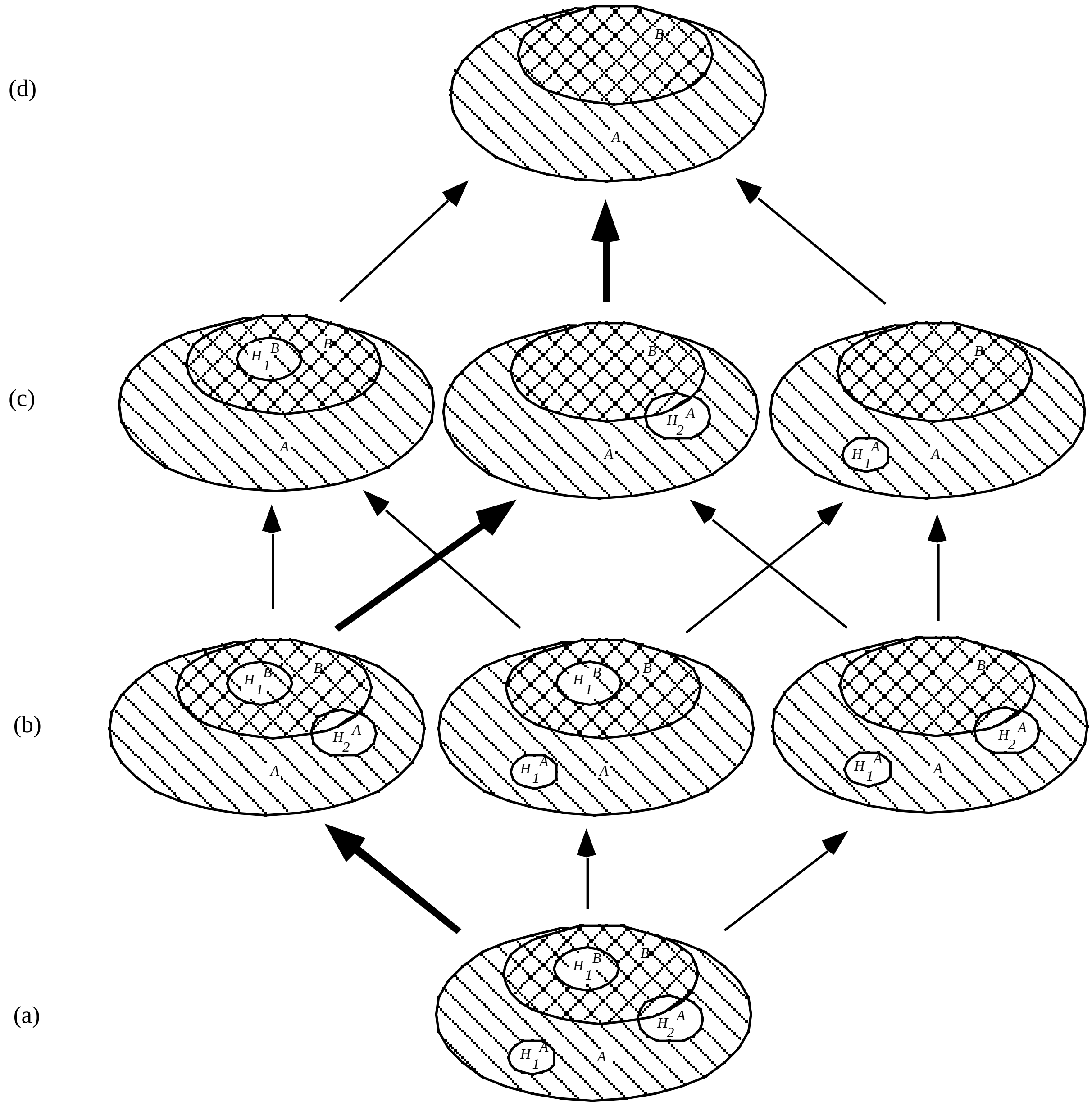


Figure 8: An example of MR levels for two regions with holes.

Tables 5a-d show the relation matrices of the four consecutive multiple-representation levels that follow the path emphasized in figure 8.

	B^*	H_1^B
A^*	covers	contains
H_1^A	disjoint	disjoint
H_2^A	overlap	disjoint

(a)

	B^*	H_1^B
A^*	covers	contains
H_2^A	overlap	disjoint

(b)

	B^*
A^*	covers
H_2^A	overlap

(c)

	B^*
A^*	covers

(d)

Table 5: The reduced relation matrices obtained from the stepwise elimination of holes H_1^A, H_1^B, and H_2^A (figure 8).

Dropping a hole from one level to the next does not affect the topological relations among the remaining objects in the scene. The process of dropping a hole corresponds in the relation matrix to deleting the corresponding row and column of that hole. The topological relation between the two generalized regions A^* and B^* allows us to express immediately in the initial scene (1) the dominant topological relation between regions A and B and (2) the topological relation that is maintained through different multiple representation levels. Also the topological relations between an object and the holes of the other object, and the topological relations between the holes of the two objects are maintained by enforcing consistency.

8. Conclusions

We presented an extension to 4-intersection to treat topological relations between objects with holes. The model is based upon the eight basic topological relations between two regions that are homeomorphic to 2-disks and applies them to the generalized region—the union of the region and its holes—and each of the region's holes. It covers topological relations (1) between two regions, each with an arbitrary number of holes, (2) between two 1-spheres, and (3) between a 1-sphere and a region (potentially with n holes).

With this model, it becomes possible to distinguish between the three different configurations in figure 2, for which the 4-intersection *per se* did not make any differences. In all three cases B^* overlaps with A's holes, however, in figure 2a A^* *covers* B^*, in figure 2b A^* *contains* B^*, and in figure 2c A^* and B^* *overlap*. Since the model uses only the topological relations that can be distinguished by the content (emptiness or non-emptiness) of the intersections, more complex topological relations may occur, which require additional topological invariants such as the component types, their dimensions, relationships with respect to the complement, and sequences (Egenhofer and Franzosa, 1993).

Although the objects' definitions excluded any separations of the interior, the method applies also to some cases of separations. The constraint for a region A with separated interiors is that its generalized region A^* must be connected. Such situations may occur when a hole, or the union of several holes, splits the interior into two or more isolated interiors.

Acknowledgments

Jayant Sharma's comments on an earlier version of this paper clarified the discussion about minimal scene relations. Also, discussions with John Herring and Bob Franzosa provided useful insight. Todd Rowell implemented a prototype of the consistency checker used here.

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