Qualitative Spatial Representation and Reasoning Techniques

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Abstract. The field of Qualitative Spatial Reasoning is now an active research area in its own right within AI (and also in Geographical Information Systems) having grown out of earlier work in philosophical logic and more general Qualitative Reasoning in AI. In this paper (which is an updated version of [25]) I will survey the state of the art in Qualitative Spatial Reasoning, covering representation and reasoning issues as well as pointing to some application areas.

1 What is Qualitative Reasoning?

The principal goal of Qualitative Reasoning (QR) [129] is to represent not only our everyday commonsense knowledge about the physical world, but also the underlying abstractions used by engineers and scientists when they create quantitative models. Endowed with such knowledge, and appropriate reasoning methods, a computer could make predictions, diagnoses and explain the behaviour of physical systems in a qualitative manner, even when a precise quantitative description is not available[1] or is computationally intractable. The key to a qualitative representation is not simply that it is symbolic, and utilises discrete quantity spaces, but that the distinctions made in these discretisations are *relevant* to the behaviour being modelled – i.e. distinctions are only introduced if they are *necessary* to model some particular aspect of the domain with respect to the task in hand. Even very simple quantity spaces can be very useful, e.g. the quantity space consisting just of $\{-, 0, +\}$, representing the two semi-open intervals of the real number line, and their dividing point, is widely used in the literature, e.g. [129]. Given such a quantity space, one then wants to be able to compute with it. There is normally a natural ordering (either partial or total) associated with a quantity space, and one form of simple but effective inference

[1] Note that although one use for qualitative reasoning is that it allows inferences to be made in the absence of complete knowledge, it does this not by probabilistic or fuzzy techniques (which may rely on arbitrarily assigned probabilities or membership values) but by refusing to differentiate between quantities unless there is sufficient evidence to do so; this is achieved essentially by collapsing 'indistinguishable' values into an equivalence class which becomes a qualitative quantity. (The case where the indistinguishability relation is not an equivalence relation has not been much considered, except by [86, 83].)

is to exploit the transitivity of the ordering relation. More interestingly, one can also devise qualitative arithmetic algebras [129]; for example one can perform addition on the above qualitative quantity space and add '+' to '+' to get '+'; however certain operations will in general yield ambiguous results (e.g. adding '+' and '−' yields no information). This is a recurring feature of Qualitative Reasoning – not surprisingly, reducing the precision of the measuring scale decreases the accuracy of the answer. Much research in the Qualitative Reasoning literature is devoted to overcoming the detrimental effects on the search space resulting from this ambiguity, though there is not space here to delve into this work. However one other aspect of the work in traditional Qualitative Reasoning is worth noting here: a standard assumption is made that change is continuous; thus, for example, in the quantity space mentioned above, a variable cannot transition from − to + without first taking the value 0. We shall see this idea recurring in the work on qualitative spatial reasoning described below.

2 What is Qualitative Spatial Reasoning?

QR has now become a mature subfield of AI as evidenced by its 11th annual international workshop, several books (e.g. [129] [51],[88]) and a wealth of conference and journal publications. Although the field has broadened to become more than just Qualitative Physics (as it was first known), the bulk of the work has dealt with reasoning about scalar quantities, whether they denote the level of a liquid in a tank, the operating region of a transistor or the amount of unemployment in a model of an economy.

Space, which is multidimensional and not adequately represented by single scalar quantities, has only a recently become a significant research area within the field of QR, and, more generally, in the Knowledge Representation community. In part, this may be due to the *Poverty Conjecture* promulgated by Forbus, Nielsen and Faltings [129]: "there is no purely qualitative, general purpose kinematics". Of course, qualitative spatial reasoning (QSR) is more than just kinematics, but it is instructive to recall their third (and strongest) argument for the conjecture – "No total order: quantity spaces don't work in more than one dimension, leaving little hope for concluding much about combining weak information about spatial properties". They correctly identify transitivity of values as a key feature of a qualitative quantity space but doubt that this can be exploited much in higher dimensions and conclude: "we suspect the space of representations in higher dimensions is sparse; that for spatial reasoning almost nothing weaker than numbers will do".

The challenge of QSR then is to provide calculi which allow a machine to represent and reason with spatial entities of higher dimension, without resorting to the traditional quantitative techniques prevalent in, for example, the computer graphics or computer vision communities.

Happily, over the last few years there has been an increasing amount of research which tends to refute, or at least weaken the 'poverty conjecture'. There is a surprisingly rich diversity of qualitative spatial representations addressing many different aspects of space including topology, orientation, shape, size and distance; moreover, these can exploit transitivity as demonstrated by the relatively sparse transitivity tables (cf the well known table for Allen's interval temporal logic [129]) which have been built for these representations (actually 'composition tables' is a better name for these structures, as explained below).

In the remainder of this paper, first I will mention some possible applications of QSR, then I will survey the main aspects of the representation of qualitative spatial knowledge including ontological aspects, topology, distance, orientation, shape and uncertainty. Then I will move on to qualitative spatial reasoning including reasoning about spatial change. The paper concludes with a discussion of theoretical results and a glimpse at future work. This paper is a revised and updated version of [25]. Although I have tried to cover the main areas of QSR, this paper is certainly not a comprehensive survey of the subject and there is much interesting work which unfortunately I have not had space to describe here.

3 Possible applications of qualitative spatial reasoning

Researchers in qualitative spatial reasoning are motivated by a wide variety of possible application areas, including: Geographical Information Systems (GIS), robotic navigation, high level vision, the semantics of spatial prepositions in natural languages, engineering design, commonsense reasoning about physical situations, and specifying visual language syntax and semantics. Below I will briefly discuss each of these areas, arguing the need for some kind qualitative spatial representation. Other application areas include document-type recognition [56] and domains where space is used as a metaphor, e.g. [90], [104].

GIS are now commonplace, but a major problem is how to interact with these systems: typically, gigabytes of information are stored, whether in vector or raster format, but users often want to abstract away from this mass of numerical data, and obtain a high level symbolic description of the data or want to specify a query in a way which is essentially, or at least largely, qualitative. Arguably, the next generation of GIS will be built on concepts arising from *Naive Geography* [47] which requires a theory of qualitative spatial reasoning.

Although robotic navigation ultimately requires numerically specified directions to the robot to move or turn, this is not usually the best way to plan a route or other spatially oriented task: the AI planning literature [123] has long shown the effectiveness of hierarchical planning with detailed decisions (e.g. about how or exactly where to move) being delayed until a high level plan has been achieved; moreover the robot's model of its environment may be imperfect (either because of inaccurate sensors or because of lack of information), leading to an inability to use more standard robot navigation techniques. A qualitative model of space would facilitate planning in such situations. One example of this kind of work is [89]; another, solving the well known 'piano mover's problem' is [50].

While computer vision has made great progress in recent years in developing low level techniques to process image data, there is now a movement back (e.g. [52]) to try to find more symbolic techniques to take the results of these low level computations and produce higher level descriptions of the scene or video input; often (part of) what is required is a description of the spatial relationship between the various objects or regions found in the scene; however the predicates used to describe these relationships must be sufficiently high level, or qualitative, in order to ensure that scenes which are semantically close have identical or at least very similar descriptions.

Perhaps one of the most obvious domains requiring some kind of theory of qualitative spatial representation is the task of finding some formal way of describing the meaning of natural language spatial prepositions such as "inside", "through", "to the left of" etc. This is a difficult task, not least because of the multiple ways in which such prepositions can be used (e.g. [82] cites many different meanings of "in"); however at least having a formal language at the right conceptual level enables these different meanings to be properly distinguished. Examples of research in this area include [4, 128].

Engineering design, like robotic navigation, ultimately normally requires a fully metric description; however, at the early stages of the design process, it is often better to concentrate on the high level design, which can often be expressed qualitatively. The field of qualitative kinematics (e.g. [49]) is largely concerned with supporting this kind of activity.

The fields of qualitative physics and naive physics [129] have concerned themselves with trying to represent and reason about a wide variety of physical situations, given only qualitative information. Much of the motivation for this was given above in the section on qualitative reasoning; however traditionally these fields, in particular qualitative physics, have had a rather impoverished spatial capacity in their representations, typically restricting information to that which can be captured along a single dimension; adding a richer theory of qualitative spatial reasoning to these fields would increase the class of problems they could tackle.

Finally, the study and design of visual languages, either visual programming languages or some kind of representation language, perhaps as part of a user interface, has become rather fashionable; however, many of these languages lack a formal specification of the kind that is normally expected of a textual programming or representation language. Although some of these visual languages make metric distinctions, often they are predominantly qualitative in the sense that the exact shape, size, length etc. of the various components of the diagram or picture are unimportant – rather, what is important is the topological relationship between these components and thus a theory of qualitative spatial representation may be applicable in specifying such languages [65, 64, 77, 78].

4 Aspects of qualitative spatial representation

There are many different aspects to space and therefore to its representation: not only do we have to decide on what kinds of spatial entity we will admit (i.e. commit to a particular ontology of space), but also we can consider developing different kinds of ways of describing the relationship between these kinds of spatial entity; for example we may consider just their topology, or their sizes or the distance between them, or their shape. Of course, these notions are not entirely independent as we shall see below.

4.1 Ontology

In developing a theory of space, one can either decide that one will create a *pure* theory of space, or an *applied* one, situated in the intended domain of application; the question is whether one considers aspects of the domain, such as rigidity of objects, which would prevent certain spatial relationships, such as interpenetration, from holding. In order to simplify matters in this paper, we shall concentrate mainly on pure spatial theories – one could very well argue that such a theory should necessarily precede an applied one which would be obtained by extending a purely spatial theory.

Traditionally, in mathematical theories of space, points are considered as primary primitive spatial entities (or perhaps points and lines), and extended spatial entities such as regions are defined, if necessary, as sets of points. However, within the QSR community, there has been a strong tendency to take regions of space as the primitive spatial entity. There are several reasons for this. If one is interested in using the spatial theory for reasoning about physical objects, then one might argue that the spatial extension of any actual physical object must be region-like rather than a lower dimensional entity. Similarly, most natural language (non mathematical) uses of the word "point" do not refer to a mathematical point: consider sentences such as "the point of pencil is blunt". Moreover, it turns out that one can define points, if required, from regions (e.g. [11] following earlier work [16, 130]). Another reason against taking points as primitive is that many people find it counterintuitive that extended regions can be composed entirely of dimensionless points occupying no space! However, it must be admitted that sometimes it is useful to make an abstraction and view a 3D physical entity such as a potholed road as a 2D or even 1D entity. Of course, once entities of different dimensions are admitted, a further question arises as to whether mixed dimension entities are to be allowed. Further discussion of this issue can be found in [27, 73, 26].

Another ontological question is what is the nature of the embedding space, i.e. the universal spatial entity? Conventionally, one might take this to be R^n for some n, but one can imagine applications where discrete (e.g. [43]), finite (e.g. [72], or non convex (e.g. non connected) universes might be useful.

Once one has decided on these ontological questions, there are further issues: in particular, what primitive "computations" will be allowed? In a logical theory, this amounts to deciding what primitive non logical symbols one will admit without definition, only being constrained by some set of axioms. One could argue that this set of primitives should be small, not only for mathematical elegance and to make it perhaps easier to assess the consistency of the theory, but also because this will simplify the interface of the symbolic system to a perceptual component resulting in fewer primitives to be implemented; the converse argument might be that the resulting symbolic inferences may be more complicated (and thus perhaps slower) and for the kinds of reasons argued for in [79], i.e. that rather than just a few primitives it is more natural to have a large and rich set of concepts which are given meaning by many axioms which connect them in many different ways.

One final ontological question we will mention here is how to model the multi dimensionality of space? One approach (which might appear superficially attractive) is to attempt to model space by considering each dimension separately, projecting each region to each of the dimensions and reasoning along each dimension separately; however, this is easily seen to be inadequate: e.g. two individuals may overlap when projected to both the x and y axes individually, when in fact they do not overlap at all.

4.2 Topology

Topology is perhaps the most fundamental aspect of space and certainly one that has been studied extensively within the mathematical literature. It is often described informally as "rubber sheet geometry", although this is not quite accurate. However, it is clear that topology must form a fundamental aspect of qualitative spatial reasoning since topology certainly can only make qualitative distinctions; the question then arises: can one not simply import a traditional mathematical topological theory wholesale into a qualitative spatial representation? Although various qualitative spatial theories have been influenced by mathematical topology, there are a number of reasons why such a wholesale importation seems undesirable in general [73]; not only does traditional topology deal with much more abstract spaces that pertain in physical space or the space to be found in the kinds of applications mentioned above, but also we are interested in qualitative spatial *reasoning* not just representation, and this has been paid little attention in mathematics and indeed since typical formulations involve higher order logic, no reasonable computational mechanism would seem to be immediately obvious.

One exception to the disregard of earlier topological theories by the QSR community, is the tradition of work to be found in the philosophical logic literature, e.g. [131, 36, 132, 15, 16, 11]. This work has built axiomatic theories of space which are predominantly topological in nature, and which are based on taking regions rather than points as primitive – indeed, this tradition has been described as "pointless geometries" [61]. In particular the work of Clarke [15, 16] has lead to the development of the so called RCC systems [109, 108, 107, 105, 34, 28, 7, 68, 24, 73, 27, 26] and has also been developed further by [128, 3].

Clarke took as his primitive notion the idea of two regions x and y being connected (sharing a point, if one wants to think of regions as consisting of sets of points): $\mathsf{C}(x, y)$. In the RCC system this interpretation[2] is slightly changed to the closures of the regions sharing a point[3] – this has the effect of collapsing the

[2] A formal semantics for RCC has been given by [69, 37, 121].

[3] Actually, given the disdain of the RCC theory as presented in [108] for points, a

distinction between a region, its closure and its interior, which it is argued has no relevance for the kinds of domain with which QSR is concerned (another reason for abandoning traditional mathematical topology). This primitive is surprisingly powerful: it is possible to define many predicates and functions which capture interesting and useful topological distinctions. The set of eight jointly exhaustive and pairwise disjoint (JEPD) relations illustrated in figure 1 are one particularly useful set (often known as the RCC8 calculus) and indeed have been defined in an entirely different way by [42] – see below.

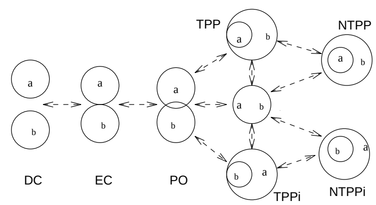

Fig. 1. 2D illustrations of of the relations of the RCC8 calculus and their continuous transitions (*conceptual neighbourhood*).

The work of [128, 3] mentioned above is also based on Clarke's calculus. The original interpretation of $\mathsf{C}(x,y)$ is retained though the general fusion operator is discarded, it is made first order and several mistakes are corrected. An additional predicate $\mathsf{WC}(x,y)$ is defined in order to try to model the distinction between two bodies being 'joined' and merely touching – consider the left and right halves of a table top compared to the table top and a book resting on it: the former case is modelled by EC(lefthalf,righthalf)[4] whilst the latter by WC(book,tabletop). $\mathsf{WC}(x,y)$ is true when x is connected to the closure of the *topological neighbourhood* of y, i.e. the smallest open region the closure of y is part of.

Expressiveness of $\mathsf{C}(x,y)$ The predicate $\mathsf{C}(x,yy)$ can be used to define many more predicates than simply the RCC8 relations and $\mathsf{WC}(x,y)$. For example one could define predicates which counted the number of times two regions touched. In a series of papers, [67, 68, 73, 71], Gotts sets himself the task of distinguishing a 'doughnut' (a solid, one-piece region with a single hole). It is shown how (given certain assumptions about the universe of discourse and the kinds of regions inhabiting it) all the shapes depicted in Fig.2 can be distinguished. In so doing he defines many predicates in terms of the $\mathsf{C}(x,y)$ primitive, for example the distinction between being a firm and non firm tangential part (FTPP), i.e. whether the tangential connection is point-like or not. Fig.3 illustrates another

better interpretation, given some suitable distance metric, would be that $\mathsf{C}(x,y)$ means that the distance between x and y is zero, c.f. [121].

[4] And thus C(lefthalf,righthalf) holds too.

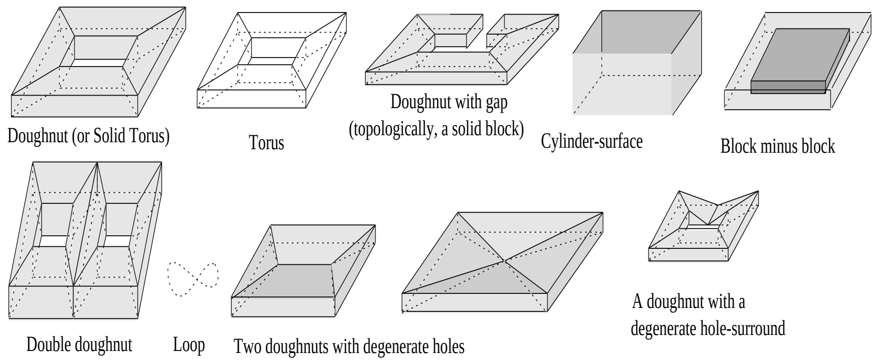

Fig. 2. It is possible to distinguish all these shapes using $\mathsf{C}(x, y)$ alone.

range of topological distinctions between one-piece (CON) regions that can be made (under certain assumptions) using C. A region, if it is connected, may or may not also be interior-connected (INCON), meaning that the interior of the region is all one piece. It is relatively easy to express this property (or its converse) in RCC terms. However, $\mathsf{INCON}(r)$ does not rule out all regions with anomalous boundaries, and in particular does not exclude the region at the right of Fig.3, nor any of the final three cases illustrated in Fig.2, which do have one-piece interiors, but which nevertheless have boundaries which are *not* (respectively) simple curves or surfaces, having 'anomalies' in the form of points which do not have line-like (or disc-like) neighbourhoods within the boundary (i.e. which are *locally Euclidean.*)

It appears possible using $\mathsf{C}(x, y)$ to define [68] a predicate (WCON) that will rule out the INCON but anomalous cases of Fig.3, but it is by no means straightforward,[5] and it is not demonstrated conclusively in [68] that the definitions do what is intended. One source of the difficulties arising is the fact that within RCC, since all regions in a particular model of the axioms are of the same dimensionality as the universal region, u , assuming u itself to be of uniform dimensionality (this follows from the fact that all regions have an NTPP), there is no way to refer directly to the boundary of a region or to the dimensionality of the shared boundary of two EC regions, or to any relations between entities of different dimensionalities.

[5] Note, however, that this task becomes almost trivial once the $\mathsf{conv}(x)$ primitive is introduced in Section 4.3.

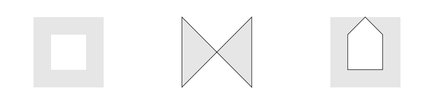

Fig. 3. Types of CON Region

In cases where reasoning about dimensionality becomes important, RCC and related systems based on a $\mathsf{C}(x, y)$ predicate are not very powerful (and to reason about regions of different dimensionality is impossible with out imposing a sort structure and essentially taking a copy of the theory for each dimension-sort). To remedy this Gotts proposed a new primitive $\mathsf{INCH}(x, y)$, whose intended interpretation is that spatial entity x *includes a chunk of* y, where the included chunk is of the same dimension as x. The two entities may be of differing (though uniform) dimension. Thus if x is line crossing a 2D region y, then $\mathsf{INCH}(x, y)$ is true, but not vice versa. It is easy to define $\mathsf{C}(x, y)$ in terms of INCH, but not vice versa, so the previous RCC system can be defined as a sub theory. An initial exposition of this theory can be found in [70].

Another proposal addressing the problem of representing and reasoning about regions of differing dimensionality (though still not of mixed dimensionality) is [59]. Here, two primitives are proposed, the mereological part relation, $\mathsf{P}(x, y)$, and a boundary operator, $\mathsf{B}(x, y)$ – x is the boundary of y (being a region of one less dimension). This follows on from other theories which introduce boundaries of regions explicitly (e.g. [119, 120, 125, 109]) but which did not explicitly introduce dimensional reasoning.

Topology via "n-intersections" An alternative approach to representing and reasoning about topological relations has been promulgated via series of papers (e.g.[23, 39, 41, 41, 40, 46, 42]). In the most recent calculus three sets of points are associated with every region – its interior, boundary and complement; the relationship between two regions can be characterized by a 3x3 matrix,[6] called the 9-intersection, each of whose elements denotes whether the intersection of the corresponding sets from each region are empty or not. Although it would seem that there are $2^9 = 512$ possible matrices, after taking into account the physical reality of 2D space and some specific assumptions about the nature of regions, which can then be translated into constraints between the matrix values, it turns out that there are exactly 8 remaining matrices, corresponding to the eight RCC8 relations. One can use this calculus to reason about regions which have holes by classifying the relationship not only between each pair of regions, but also the relationship between each hole of each region and the other region and each of its holes [45]. By changing the underlying assumptions about what a region is, and by allowing the matrix to represent the codimension of the intersection, different calculi with more JEPD relations can be derived. For example, one may derive a calculus for representing and reasoning about regions in Z^2 rather than R^2 [43] – there are 16 possible matrices representing the set of JEPD relations in this case. Alternatively, one can extend the representation by noting in each matrix cell the dimension of the intersection rather than simply whether it exists or not [17]; this allows one to enumerate all the relations between areas, lines and points –

[6] Actually, a simpler 2x2 matrix [41], known as the 4-intersection, featuring just the interior and boundary is sufficient to describe the eight RCC relations; however the 3x3 matrix allows more expressive sets of relations to be defined as noted below since it takes into account the relationship between the region and its embedding space.

this extension is known as the "dimension extended method (DEM)". [22] have noted the very large number of possible relationships that may be defined in this way and have proposed a way (which they call the "calculus based method (CBM)", to generate all these from a set of five polymorphic binary relations between a pair of spatial entities x and y: disjoint, touch (a/a, l/l, l/a, p/a, p/l), in, overlap (a/a, l/l), cross (restrictions on the arguments are denoted by the notation α/β, e.g. a/a meaning that both arguments must be areal, p/p that they must be points and l/l that they must both be linear). In addition, operators are introduced to denote the boundary of a region and the two endpoints of a non circular line. A complex relation between x and y may then formed by conjoining atomic propositions formed by using one of the five relations above, whose arguments may be either be x or y or a boundary or endpoint operator applied to x or y. [22] have analysed the number of JEPD relations relations) for each of the techniques mentioned above (4- and 9-intersections, DIM and CBM). For the most expressive calculus (either the CBM or the combination of the 9-intersection and the DIM), there are 9 area/area relations, 31 line/area relations, 3 point/area relations, 33 line/line relations, 3 point/line relations and 2 point/point relations giving a grand total of 81.

Mereology and Topology Although mereology (being the theory of the part-whole relationship) would seem at first sight simply to be a subtheory of topology (and indeed is presented thus in the topological theories mentioned so far in this section), there are arguments against this view. Varzi [126] has discussed the issue and notes that whilst certain mereology is not sufficient by itself, there are three main ways in which theories in the literature have proposed integrating topology and mereology:

1. Generalise mereology by adding a topological primitive. This is the approach taken by, for example, [12] who add the topological primitive $\mathsf{SC}(x)$, i.e. x is a self connected (one-piece) spatial entity to the mereological part relation. Alternatively a single primitive can be used to as in [125]: "x and y are connected parts of z. Generally, this approach forces the existence of boundary elements (i.e. spatial entities of lower dimensions). The main advantage of separate theories of mereology and topology is that it allows colocation without sharing parts which is not easily possible in the second two approaches below.
2. Topology is primal and mereology is a sub theory. For example in the topological theories based on $\mathsf{C}(x, y)$, such as those mentioned above, one defines $\mathsf{P}(x, y)$ from $\mathsf{C}(x, y)$. This has the elegance of being a single unified theory, but colocation implies sharing of parts. These theories are normally boundaryless (i.e. without lower dimensional spatial entities) but this is not absolutely necessary [109, 70]. Thus, for example $\mathsf{EC}(x, y)$ not necessarily explained by sharing a boundary.
3. The final approach is that taken by [48], i.e. topology is introduced as a specialised domain specific sub theory of mereology. Of course an additional primitive needs to be introduced since mereology alone is not powerful

enough to define topology. The idea is to use restricted quantification by introducing a sortal predicate $\mathsf{Region}(x)$. $\mathsf{C}(x,y)$ can then be defined thus: $\mathsf{C}(x,y) \equiv_{\text{def}} \mathsf{O}(x,y) \wedge \mathsf{Region}(x) \wedge \mathsf{Region}(y)$.

4.3 Between Topology and Fully Metric Spatial Representation

Topology can be seen as perhaps the most abstract and most qualitative spatial representation, furthest removed from fully metric representations. However it is clear that although potentially useful there many be many domains where topological information alone is insufficient but it would still be desirable to have a qualitative representation. In the following subsections a selection of different ways of add qualitative non topological information are presented.

Orientation Orientation is a naturally qualitative property: in 2D it is very common to talk about clockwise or anticlockwise orientation for instance. However, unlike most of the topological relations on spatial entities mentioned above, orientation is not a binary relation – at least three elements need to be specified to give an orientation between two of them (and possibly more in dimensions higher than 2D). If we want to specify the orientation of a *primary object* (PO) with respect to a *reference object* (RO), then we need some kind of *frame of reference* (FofR). An *extrinsic* frame of reference imposes an external, immutable orientation: e.g. gravitation, a fixed coordinate system, or an third object (such as the North pole). A *deictic* frame of reference is with respect to the "speaker" or some other internal observer. Finally, an *intrinsic* frame of reference exploits some inherent property of the RO – many objects have a natural "front", e.g. humans, buildings and boats. This categorization manifests itself in the display of qualitative orientation calculi to be found in the literature: certain calculi have an explicit triadic relation while others presuppose an extrinsic frame of reference and, for example, use compass directions [54, 80]. Of those with explicit triadic relations is it especially worth mentioning the work of Schlieder [114] (following earlier work [66]) who develops a calculus based on a function which maps triples of points to one of three qualitative values, + , 0 or -, denoting anticlockwise, colinear and clockwise orientations respectively. This can be used for reasoning about visible locations in qualitative navigation tasks, or for shape description [116] or to develop a calculus for reasoning about the relative orientation of pairs of line segments [115] – see figure 4. Schlieder also notes that the notion of a *permutation sequence* [66] subsumes this framework. In this representation, given a set of points and directed lines connecting them, one chooses a new directed line l, not orthogonal to any existing line and notes the order of all the points projected onto l. One then rotates l counterclockwise until order of projection changes. As l continues to rotate, one will generate further permutations of the set of points.

Another important triadic orientation calculus is that of Roehrig [113]; this calculus is based on a relation CYCORD(x, y, z) which is true (in 2D) when x, y, z are in clockwise orientation. Roehrig shows how a number of qualitative calculi (not only orientation calculi) can be translated into the CYCORD system, whose reasoning system (implemented as a constraint logic program) can then be exploited.

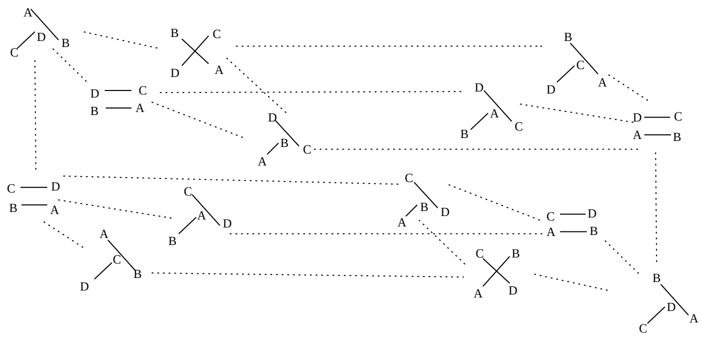

Fig. 4. The 14 JEPD relations of Schlieder's oriented line segment calculus and their *conceptual neighbourhood.*

Distance and size Distance and size are related in the sense that traditionally we use a linear scale to measure each of these aspects, even though distance is normally thought of as being a one dimensional concept, whilst size is usually associated with higher dimensional measurements such as area or volume. The domain can influence distance measurements, as we shall see below, but first I will discuss pure spatial representations. These can be divided into two main groups: those which measure on some "absolute" scale, and those which provide some kind of relative measurement. Of course, since traditional Qualitative Reasoning [129] is primarily concerned with dealing with linear quantity spaces, the qualitative algebras and the transitivity of such quantity spaces mentioned earlier can be used as a distance or size measuring representation.

Also of interest in this context are the order of magnitude calculi [95, 102] developed in the QR community. These calculi introduce measuring scales which allow one quantity to be described as being *much larger* than another, with the consequence that it requires summing many (in some formulations even an infinite number) of the former quantities in order to surpass the second, "much larger" quantity. Most of these "traditional QR" formalisms are of the "absolute" kind of representations mentioned above[7] as is the Delta calculus [134] which introduces a triadic relation,$x(>,d)y$: x is larger/bigger than y by amount d; terms such as $x(>,y)y$ mean that x is more than twice as big as y.

Of the 'relative' representations specifically developed within the spatial reasoning community, perhaps the first is the calculus proposed by de Laguna [36],

[7] Actually it is usually straightforward to specify relative measurements given an "absolute" calculus: to say that $x > y$, one may simply write $x - y = +$.

which introduces a triadic $\mathsf{CanConnect}(x, y, z)$ primitive, which is true if the body x can connect y and z by simple translation (i.e. without scaling, rotation or shape change). From this primitive it is quite easy to define notions such as equidistance, nearer than, and farther than (as well as the $\mathsf{C}(x, y)$ relation). Also note that this primitive allows a simple size metric on regions to be defined: one region is larger than another if it can connect regions that the other cannot. Another technique to determine the relative size of two objects was proposed by Mukerjee and Joe [97] and relies on being able to translate regions (assumed to be shape and size invariant) and then exploit topological relationships – if a translation is possible so that one region becomes a proper part of another, then it must be smaller. Interestingly, these seem to be about the only proposals which are grounded in a region based theory – all the other representations mentioned in this section take points as their primitive spatial entity. An interesting question arises in the case of distances between regions as to where to measure to/from – in the formalisms mentioned above the closest distance is taken, but alternatively one might be interested in the distance between centroids or some other distinguished subregion or point.

Distance is closely related to the notion of orientation: e.g. distances cannot usually be added unless they are in the same direction, and the distance between a point and region may vary depending on the orientation. Thus it is perhaps not surprising that there have been a number of calculi which are based on a primitive which combines distance and orientation information. Arguably, unless both of these aspects are represented then the calculus is not really a calculus of distance, though it might be said that this is a calculus of position rather than mere distance.

One straightforward idea [54] is to combine directions as represented by segments of the compass with a simple distance metric (*far, close*). A slightly more sophisticated idea is to introduce a primitive which defines the position of a third point with respect to a directed line segment between two other points [135] – see figure 5. A calculus which combines the Delta calculus and orientation is presented in [133].

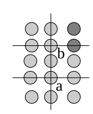

Fig. 5. There are 15 qualitatively different positions a point c (denoted by the shaded circles) can be with respect to a vector from point a to point b. Some distance information is represented, for example the darker shaded circles are in the same orientation but at different distances from ab.

The most sophisticated qualitative distance calculus to date is the framework for representing distances [81] which has been extended to include orientation[21]. In this framework a distance is expressed in a particular *frame of reference* (FofR) between a *primary object* (PO) and a *reference object* (RO). A distance system is composed of an ordered sequence of *distance relations* (between a PO and an RO), and a set of *structure relations* which give additional information about how the distance relations relate to each other (apart from their distance ordering given implicitly by the ordered sequence). Each distance has an *acceptance area* (which in the case of an isotropic space will be a region the same shape as the PO, concentrically located around the PO); the distance between successive acceptance areas defines a sequence of intervals: $\delta_1, \delta_2, \ldots$. The structure relations define relationships between these δ_i. Typical structure relations might specify a monotonicity property (the δ_i are increasing), or that each δ_i is greater than the sum of all the preceding δ_i. The structure relationships can also be used to specify order of magnitude relationships, e.g. that $\delta_i + \delta_j \sim \delta_i$ for $j < i$. The structure relationships are important in refining the *composition tables* (see below). In a *homogeneous* distance system all the distance relations have the same structure relations; however this need not be the case in a *heterogeneous* distance system. The proposed system also allows for the fact that the context may affect the distance relationships; this is handled by having different frames of reference, each with its own distance system and with inferences in different frames of reference being composed using *articulation rules* (cf. [83]). Analogously to orientation calculi, intrinsic, extrinsic and deictic frames of reference can be distinguished.

It is possible that different qualitative distance calculi (or FofR) might be needed for different scale spaces – Montello [96] suggests that there are four main kinds of scale of space, relative to the human body: *figural space* pertains to distances smaller than the human body and which thus can be perceived without movement (e.g. table top space and pictures); *vista space* is similar but pertains to spaces larger than the human body, making distortions more likely; *environmental space* cannot be perceived without moving from one location to another; finally, *geographic space* cannot be properly apprehended by moving – rather it requires indirect perception by a (figural space) map. One obvious effect of moving from one scale, or context to another, is that qualitative distance terms such as "close" will vary greatly; more subtly, distances can behave in various "non mathematical" ways in some contexts or spaces: e.g. distances may not be symmetrical – e.g. because distances are sometimes measured by time taken to travel, and an uphill journey may take longer than the return downhill journey. Distance may easily become non isotropic when time taken to travel is used as a distance measure (i.e. travel in certain directions may take a longer time compared to the actual distance) – e.g. a fast East-West highway will tend to reduce east west travel time[81]. Another "mathematical aberration" is that in some domains the shortest distance between two points may not be a straight line (e.g. because a lake or a building might be in the way,). Human perception of distance can also be distorted – [84] reports experiments which show that cities on the west coast of the USA are viewed as being relatively closer when imagined from the east coast compared to east coast cities and vice versa when the viewpoint is changed to the other coast.

Shape As mentioned above, one can think of theories of space as forming a hierarchy ordered by expressiveness (in terms of the spatial distinctions made possible) with topology at the top and a fully metric/geometric theory at the bottom. Clearly in a purely topological theory only very limited statements can be made about the shape of a region: whether it is has holes (in the sense that a torus has a hole), or interior voids, or whether it is in one piece or not – we have already described this kind of work in section 4.2 above. [60] has observed that one can (weakly) constrain the shape of rigid objects by topological constraints using RCC8: congruent shapes can only ever be DC, EC, PO or EQ; if one shape can just fit inside the other then they can only ever be DC, EC, PO, TPP; if one shape can easily fit inside the other then they can only ever be DC, EC or PO; whilst incommensurate shapes must be DC, EC or PO.

However, if one's application demands finer grained distinctions than these, then some kind of semi-metric information has to be introduced[8]; there is a huge choice of possible primitives for extending topology with some kind of shape primitives whilst still retaining a qualitative representation (i.e. not becoming fully metric). Of course, as [20] note, the mathematical community have developed many different geometries which are less expressive than Euclidean geometry, for example projective and affine geometries, but have not necessarily developed efficient computational reasoning techniques for them[9]. The QSR community has only just started exploring the various possibilities; below we briefly describe some of the approaches.

There are a number of ways to classify these approaches; one distinction is between those techniques which constrain the possible shapes of a region and those that construct a more complex shaped region out of simpler ones (e.g. along the lines of constructive solid geometry [111], but perhaps starting from a more qualitative set of primitives). An alternative dichotomy can be drawn between representations which primarily describe the boundary of an object compared to those which represent its interior (e.g. symmetry based techniques). Arguably [13], the latter techniques are preferable since shape is inherently not a one dimensional concept.

Examples of approaches which work by describing the boundary of an object include those that classify the sequence different types of boundary segments (curving in/out, angle in/out, cusp in/out, straight) [112] or by describing the sequence of different kinds of curvature extrema[91] along its contour. Another related approach would be to pick out distinguished points on the boundary of the object (such as corners) and relate every triple of such points by using the qualitative orientation calculus described in the previous section (i.e. the shape description would consist of a sequence of -/0/+ symbols, one for each triple of distinguished points). Yet another technique is described by [85] who uses a slope projection approach to describe polygonal shape: for each corner, one

[8] Of course, the orientation and distance primitives discussed above already add something to pure topology, but as already mentioned these are largely point based and thus not directly applicable to describing region shape.

[9] Though see [5, 6].

describes whether it is convex/concave,obtuse/right-angled/acute together with a qualitative representation of the direction of the corner (chosen from a set of 9 possible values).

One approach of the latter kind is to make use of a shape abstraction primitive such as the bounding box or the convex hull. Both these techniques have been considered briefly within the n-intersection model [19] whilst the latter technique has been investigated extensively within the RCC calculus. The distinction between convex and concave regions seems fundamental to shape description. [10] RCC theory has shown that many interesting predicates can be defined once one takes the notion of a convex hull of a region (or equivalently, a predicate to test convexity) and combines it with the topological representation. By computing the topological relationships between the shape itself and the different components of the difference between the convex hull and the shape, one can distinguish many different kinds of concave shapes [24]. A refinement to this technique exploits the idea of recursive shape description [118] to describe any non convex components of the difference between the convex hull and the shape. One can also develop many sets of JEPD predicates to relate pairs of regions which directly exploit the convex hull function; such predicates give another approach to shape description: one constrains the shape of a region by specifying its relationships to other regions[33].

The convex hull is clearly a powerful primitive and in fact it has recently been shown [35] that this system essentially is equivalent to an affine geometry: any two compact planar shapes not related by an affine transformation can be distinguished by a constraint language of just $\mathsf{EC}(x)$, $\mathsf{PP}(x)$ and $\mathsf{Conv}(x)$.

Various different notions of the inside of a region can be distinguished using a convex hull primitive [24, 33] – these can all be viewed as different kinds of hole. A very interesting line of research [14, 127] has investigated exactly what holes are and proposes an axiomatisation of holes based on a new primitive: $\mathsf{Hosts}(x, y)$ – which is true if the *body* x hosts hole y; note that this is not a theory of pure space: holes cannot host other holes, only physical objects can act as hosts.

Another recent proposal [12] is to take the notion of two regions being congruent as primitive; from this it is possible to define the notion of a sphere, and then import Tarski's theory of spheres and related definitions such as 'betweenness' [122]. That this theory is more powerful than one just with convex hull is shown by the fact convexity can now be defined in a congruence based system, whilst the reverse is not the case. Also of interest in this paper is the idea of using a "grain" to eliminate small surface irregularities which might distort the shape description.

The notion of a Voronoi hull has also been used as an approach to qualitative shape description [38]. A set of voronoi regions are computed by drawing lines

[10] Note that topology only allows certain rather special kind of non convex regions to be distinguished, and in any case does not allow the concavities to be explicitly referred to – it is a theory of 'holed regions', rather than of holes per se – the distinction between "hole realist" and "irrealist" theories has been made by [14].

equidistant from each pair of closest objects under consideration. Notions such as proximity, betweenness, inside/outside, amidst can all be addressed by this technique.

Finally, before leaving the topic of shape description we should point out the work of [20] on describing shape via properties such as compactness and elongation by using the minimum bounding rectangle of the shape and the order of magnitude calculus of [95]: elongation is computed via the ratio of the sides of the minimum bounding rectangle whilst compactness by comparing the are of the shape and its minimum bounding rectangle.

4.4 Uncertainty and Vagueness

In many applications uncertainty and vagueness, for example because of indeterminate region boundaries are endemic. Such vagueness may arise for a number of reasons, perhaps because of ignorance, i.e. lack of data (e.g. sample oil well drillings) or because of temporal variation (e.g. tidal regions, a flood plain, or a river changing its course), or indeterminacy may arise because of 'field variation' (e.g. the one soil type may gradually change into another) or a region might display what one might term 'intrinsic vagueness' (e.g. 'southern England' might be so regarded since one could never agree as to what determined this region except by some arbitrary process).

Even though any qualitative calculus already makes some attempt to represent and reason about uncertainty because the qualitative abstraction hides some indeterminacy, sometimes some extra mechanism may be required. Of course, it is always possible to glue on some standard numerical technique for reasoning about uncertainty (e.g. [57]), but there has also been some research on extending existing qualitative spatial reasoning techniques to explicitly represent and reason about uncertain information. For example, a GISDATA workshop on representing and reasoning about regions with indeterminate boundaries generated two papers [31, 18] which extended the RCC calculus and the 9-intersection in very similar ways to handle these kind of regions.

The former approach, which is continued in a series of papers [30, 29, 32] postulates the existence of non crisp regions in addition to crisp regions and then adds another binary relation to RCC – x is crisper than region y. A variety of relations are then defined in terms of this primitive and this extended theory is then related to what has become known as the "egg-yolk" calculus which originated in [90] and models regions with indeterminate boundaries as a pair of regions: the 'yolk', which is definitely part of the region and the 'white', which may or may not be part of the region. It turns out that if one generalises RCC8 in this way [32] there are 252 JEPD relations between non crisp regions which can be naturally clustered into 40 sets.

The latter approach looks very similar to the egg-yolk calculus but does not consider such a fine granularity of relations; it postulates 44 JEPD relations, also clustered into groups (18 in their case) but using a more ad hoc technique to achieve this. An interesting extension to this work [19] shows that this calculus of regions with broad boundaries can be used to reason not just about regions with indeterminate boundaries but also can be specialised to cover a number of other kinds of regions including convex hulls of regions, minimum bounding rectangles, buffer zones and rasters (this last specialisation generalises the application of the n-intersection model to rasters previously undertaken by [43]).

Other approaches to spatial uncertainty are to work with an indistinguishability relation which is not transitive and thus fails to generate equivalence classes [124, 86] and the development of nonmonotonic spatial logics [117, 2].

5 Qualitative spatial reasoning

Although much of the work in QSR has concentrated on representational aspects, various computational paradigms are being investigated including constraint based reasoning (e.g. [80]). However, the most prevalent form of qualitative spatial reasoning is based on the composition table (originally known as a transitivity table [1], but now renamed since more than one relation is involved and thus it is relation composition rather than transitivity which is being represented). Given a set of n JEPD relations, the $n \times n$ composition table specifies for each pair of relations R1, and R2 such that R1(a, b) and R2(b, c) hold, what the possible relationships between a and c could be. In general, there will be a disjunction of entries, as a consequence of the qualitative nature of the calculus. Most of the calculi mentioned in this paper have had composition tables constructed for them, though this has sometimes posed something of a challenge [106]. One approach to the automatic generation of composition tables has been to try to reduce each calculus to a simple ordering relation [113]. Another, perhaps more general approach, is to formulate the calculus as a decidable theory (many calculi, e.g. the original RCC system, are presented as first order theories), ideally even as a tractable theory, and then use exhaustive theorem proving techniques to analyze and thus generate each composition table entry. A reformulation of the RCC first order theory in a zero order intuitionistic logic [11] [7] was able to generate the appropriate composition tables automatically; another approach would have been been to use a zero order modal logic [9].

Composition tables provide a very efficient form of reasoning and have certainly been the mostly commonly used form of qualitative spatial inference but they do not necessarily subsume all forms of desired reasoning. For example, reasoning with just three objects at a time will not necessarily determine all inconsistent situations in some calculi. An interesting question then arises: exactly when is composition table reasoning a sufficient inference mechanism (i.e. for which theories is it complete)[10]?

For cases when composition table based reasoning is not sufficient, then other more general constraint based reasoning may be sufficient[80, 76]; more generally one may resort to theorem proving, or preferably, some kind of specialised theorem proving system[7, 113] for example.

[11] This reformulation is interesting in that it becomes a true spatial logic, rather than a theory of space: the "logical symbols" have spatial interpretations, e.g. implication is interpreted as parthood and disjunction as the sum of two regions.

5.1 Reasoning about Spatial Change

So far we have been concerned purely with static spatial calculi, so that we can only represent and reason about snapshots of a changing world. It is thus important to develop calculi which combine space and time in an integrated fashion.

There are many kinds of spatial change: individual spatial entities may change their topological structure, their orientation, their position, their size or shape. Such changes are not necessarily independent and of course change in one spatial entity may engender a change in its spatial relationship to other entities.

Topological changes in 'single' spatial entity include: change in dimension (this is usually 'caused' by an abstraction or granularity shift rather than an 'actual' spatial change[12]; change in number of topological components (e.g. breaking a cup, fusing blobs of mercury); change in the number of tunnels (e.g. drilling through a block of wood); change in the number of interior cavities (e.g. putting a lid on a container). Such changes may also simultaneously effect changes in position, size, shape, and orientation as well as in topology (e.g. consider drilling a hole in a block of wood).

In many domains we assume that change is continuous[13], as is the case in traditional qualitative reasoning, and thus there is a requirement to build into the qualitative spatial calculus which changes in value will respect the underlying continuous nature of change, and this requirement is of course common to all the different kinds of spatial change we have mentioned above. It is thus important to know which qualitative values or relations are *neighbours* in the sense that if a value or predicate holds at one time, then there is some continuous change possible such that the next value or predicate to hold will be a neighbour. Continuity networks defining such neighbours are often called *conceptual neighbourhoods* in the literature following the use of the term [55] to describe the of structure Allen's 13 JEPD temporal relations [1] according to their conceptual closeness (e.g. *meets* is a neighbour of both *overlaps* and *before*). Most of the qualitative spatial calculi reported in this paper have had conceptual neighbourhoods constructed for them; see figures 1 and 4 for example[14].

Perhaps the most common form of computation in the traditional QR literature is qualitative simulation; using conceptual neighbourhood diagrams is quite easy to build a qualitative *spatial* simulator [34]. Such a simulator takes a set of ground atomic statements describing an initial state[15] and constructs a tree of

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[12] E.g. we may view a road as being a 1D line on a map, a 2D entity when we consider whether it is wide enough for an outsize load, and a 3D entity as we consider the range of mountains it passes over, or the potholes and a particularly delicate cargo.

[13] Sometimes changes are discontinuous, e.g. when political fiat moves the boundaries of geopolitical entities in a discontinuous manner.

[14] A close related notion is that of "closest topological distance" [44] – two predicates are neighbours if their respective n-intersection matrices differ by fewer entries than any other predicates; however the resulting neighbourhood graph is not identical to the true conceptual neighbourhood or continuity graph – some links are missing.

[15] The construction of an envisioner [129] rather than a simulator would also be possible

future possible states – the branching of the tree results from the ambiguity of the qualitative calculus. Of course, continuity alone does not provide sufficient constraints to restrict the generation of next possible states to a reasonably small set in general – domain specific constraints are required in addition. These may be of two kinds: intra state constraints restrict the spatial relationships that may hold within any state whilst inter state constraints restrict what can hold between adjacent states (or in general, across a sequence of states). Both of these constraint types can be used to prune otherwise acceptable next states from the simulation tree. Additional pruning is required to make sure that each state is consistent with respect to the semantics of the calculus (e.g. that there is no cycle of proper part relationships) – the composition table may be used for this purpose.

A desirable extension, by analogy with earlier QR work, would be to incorporate a proper theory of spatial processes couched in a language of QSR; some work in this direction is reported in: [93] who considers a field based theory of spatial processes such as heat flow; [44] who consider which traversals of their version of the conceptual neighbourhood diagram for an 8 relation topological calculus analogous to RCC8 correspond to processes such as expansion of a region, rotation of region etc; [91] considers how the processes of protrusion and resistance cause changes in his boundary based shape description language mentioned in section 4.3 above – given two shapes he can then infer sequences of processes which could cause one to change into the other. Also worthy of note is the qualitative spatial simulation work of [103] based on the QSIM system [129].

One problem is that the conceptual neighbourhood is usually built manually for each new calculus – a sometimes arduous and error prone operation if there are many relations; techniques to derive these automatically would be very useful. An analysis of the structure of conceptual neighbourhoods is reported by Ligozat [92] goes some way towards this goal. A more foundational approach which exploits the continuity of the underlying semantic spaces has been investigated by [58] – this analysis not only allows the construction of a conceptual neighbourhood for a class of relations from a semantics, but also infers which relations *dominate* other relations: a relation R_1 dominates R_2 if R_2 can hold over an interval followed/preceded by R_1 instantaneously. E.g. in RCC8 TPP dominates NTPP and PO, while EQ dominates all of its neighbouring relations. Dominance is analogous to the equality change law to be found in traditional QR [129] and allows a stricter temporal order to be imposed on events occurring in a qualitative simulation.

5.2 Theoretical results in QSR

There are a number of theoretical questions of interest. Not all calculi have been given a formal semantics by their inventors and even for those that have there is the question of whether it is the best or simplest semantics. Given a semantics one can ask whether the task of showing a set of formulae is consistent or whether

of course. See also the transition calculus approach of [63].

one set entails another is decidable, and if it is what is the complexity of the decision procedure. One can ask if a theory is complete, either in the weak sense of every true formula being provable, or the stronger sense of whether every formula is made either true or false in the theory. Obviously, complete first order theories are also decidable. Finally, there is the property of being categorical, i.e. whether all models are isomorphic? Since theories may have both finite and infinite models, a more interesting property is $\aleph_0$ categoricity, i.e. whether all infinite models are isomorphic.

[100] set out to answer the question as to whether there is something special about region based theories from the ontological viewpoint? They believe the answer is in the negative, at least for 2D mereotopology: they show, under certain assumptions, that the standard 2D point based interpretation is simplest model (prime model) proved under assumptions; the only alternative models involve regions with infinitely many pieces. But it may be argued, that it is still useful to have region based theories even if they are always interpretable point set theoretically.

A fundamental result on decidability which has widespread applicability in qualitative spatial theories is that of [75] which shows that although of course Boolean algebra is decidable, adding either a closure operation or an external connection relation results in an undecidable system since one can then encode arbitrary statements of arithmetic. This implies that Clarke's calclus and all the related calculi such as the first order theory of RCC, and the calculi of [3] and [12] are all undecidable.

The question then becomes whether there are any decidable subsystems? [16] The constraint language of RCC8 has been shown to be decidable [7] – this was achieved by encoding each RCC8 relation as a set of formulae in intuitionistic propositional calculus which is a decidable calculus. This language was subsequently shown to be tractable [98] – in fact the satisfaction problem is solvable in polylogarithmic time since it is in the complexity class NC. However the constraint language of 2^{RCC8} (i.e. where constraints may be arbitrary disjunctions of RCC8 relations) is not tractable, though some subsets are tractable – [110] have identified a maximal tractable subset of the constraint language of 2^{RCC8} and furthermore have shown that for path consistency is sufficient for deciding consistency in this case. As in the case of identifying the maximal tractable subset of Allen's interval calculus [99], the analysis relies on an exhaustive computer generated case analysis. Also of interest is the analysis of [74] which considers an RCC8-like calculus and two simpler calculi and determines which of a number of different problem instances of relational consistency and planar realizability are tractable and which are not – the latter is the harder problem. It has also been shown that the constraint language of $\mathsf{EC}(x)$, $\mathsf{PP}(x)$ and $\mathsf{Conv}(x)$ is intractable (it is at least as hard as determining whether a set of algebraic constraints over the reals is consistent) [35].

Clarke's system has been given a semantics (regular sets of Euclidean space

[16] Rather in the same manner as the description logic community have sought to find the line dividing decidability from undecidability and tractability from intractability.

are models) and has been shown to be complete in the weak sense [11]. Unfortunately it turns out that contrary to Clarke's intention, only mereological relations are expressible! The theory in fact characterises complete atomless Boolean algebra. The system of [3] which corrects the problems in Clarke as mentioned above, is given a semantics and shown to be complete by the authors but their inclusion of the notion of 'weak connection' forces a non standard model since models must be non dense. [17]

A completeness result (in the strong sense) has been derived by [101] who give a complete 2D topological theory whose elements are 2D finite (polygonal) regions and whose primitives are: the null and universal regions, the Boolean functions (+,*,−), and a predicate to test for a region being one piece. The theory is first order but requires an infinitary rule of inference (which is not surprising in view of the undecidability of first order topology mentioned above [75]. The infinitary rule of inference guarantees the existence of models in which every region is sum of finitely many connected regions. The resulting theory is complete but not decidable.

Notwithstanding the attempt [8] to derive a complete first order topological theory, it is now clear that no first order finite axiomatisation of topology can be complete or categorical because it is not decidable.

6 Final comments

An issue which has not been much addressed yet in the QSR literature is the issue of cognitive validity – claims are often made that qualitative reasoning is akin to human reasoning, but with little or no empirical justification; one exception to this work is the study made of a calculus for representing topological relations between regions and lines [94] where native speakers of several different languages were asked to perform tasks in which they correlated spatial expressions such as "the road goes through the park" with a variety of diagrams which depicted a line and a region which the subjects were told to interpret as as road and a park. Another study is [87] which has investigated the preferred Allen relation (interpreted as a 1D spatial relation) in the case that the composition table entry is a disjunction. Perhaps the fact that humans seem to have a preferred model explains why they are able to reason efficiently in the presence of the kind of ambiguity engendered by qualitative representations.

As in so many other fields of knowledge representation it is unlikely that a single universal spatial representation language will emerge – rather, the best we can hope for is that the field will develop a library of representational and

[17] This enforced abandonment of R^n as a model leads one to question whether it is indeed a good idea to try to model the proposed distinction between strong and weak connection topologically in a purely spatial theory, rather than in an applied theory of physical bodies and material substances together with the regions they occupy. It should be pointed out that they do propose an extension to their theory in which they allow the spatial granularity to be varied; as finer and finer granularities are considered, so fewer instances of $\mathsf{WC}(x, y)$ are true and in the limit the theory tends to the classical topological model.

reasoning devices and some criteria for their most successful application. Moreover, as in the case of non spatial qualitative reasoning, quantitative knowledge and reasoning must not be ignored – qualitative and quantitative reasoning are complementary techniques and research is needed to ensure they can be integrated, for example by developing reliable ways of translating between the two kinds of formalism[18]. Equally, interfacing symbolic QSR to the techniques being developed by the diagrammatic reasoning community [62] is an interesting and important challenge.

In this paper I have tried to provide an overview of the field of qualitative spatial reasoning; however the field is active and there has not been space to cover everything (for example qualitative kinematics [49]). A European funded *Human Capital and Mobility* Network, Spacenet, links together eleven sites working in the field of qualitative spatial reasoning and the web page (http://www.scs.leeds.ac.uk/spacenet/) provides an entry to point to the ongoing work at these sites and elsewhere. Other relevant web sites include the spatial reasoning home page at http://www.cs.albany.edu/~amit/bib/spatsites.html and the spatio-temporal home page at: http://www.cs.aukland.ac.nz/~hans/spacetime/. An online searchable web bibliographies can be found at http://www.cs.albany.edu/~amit/bib/spatial.html.

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[18] Some existing research on this problem includes [53, 52].

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