Spatial Cognition and Computation **2:** 435–466, 2000.

Approximate qualitative spatial reasoning

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Received 27 March 2001; accepted 07 March 2002

Abstract. Qualitative relations between spatial regions play an important role in the representation and manipulation of spatial knowledge. The RCC5 and RCC8 systems of relations, used in the Region-Connection Calculus, are of fundamental importance. These two systems deal with ideal regions having precisely determined location. However, in many practical examples of spatial reasoning, regions are represented by finite approximations rather than known precisely. Approximations may be given by describing how a region relates to cells forming a partition of the space under consideration. Although the RCC5 and RCC8 systems have been generalized to "egg-yolk" regions, in order to model certain types of vagueness, their extension to regions approximated in this way has not been discussed before. This paper presents two methods, the syntactic and the semantic, by which the RCC5 and RCC8 systems may be defined for approximate regions. The syntactic uses algebraic operations on approximate regions which generalize operations on precise regions. The semantic method makes use of the set of precise regions which could be the intended interpretation of an approximate region. Relationships between these two methods are discussed in detail.

Key words: approximate spatial reasoning, qualitative spatial reasoning, topological relations

1. Introduction

Our knowledge of the spatial world is necessarily approximate. However accurate our measuring instruments, any description of the distance between two places can only be given to some finite resolution. In determining such a distance there are two problems: places in the real world are not points, and even if we could associate locations with ideal points, it would be impossible to measure the distance between the points to an infinite level of precision. The idealized view of spatial data given by coordinates expressed in terms of the real numbers is at variance with reality on two counts. Firstly, it is not possible to relate a cartesian frame of reference to the geographical world in a wholly accurate way; we cannot isolate a "point" on the globe and assert it is the origin of the coordinate system. All that can be physically identified is a region. Admittedly we can determine very small regions but regions, however small, contain infinitely many ideal points. Secondly, we cannot relate features in the world to the frame of reference in a wholly accurate way. Even if we could fix the frame, we cannot specify with infinite precision the distances of a location from the coordinate axes.

Despite these accepted and well known difficulties, it is still conventional to use the mathematically ideal real numbers as a basis for the description and manipulation of spatial data. Conventional geographic information systems (GIS) are based on coordinate systems expressed in reals and much work proceeds on the basis that should more detail be required it is always possible to add more figures after the decimal point.

In response to the unreasonable accuracy pretended by coordinate descriptions based on the reals, and also to the observation that humans manipulate spatial data without apparent recourse to such descriptions, various proposals have been advanced for spatial descriptions which are qualitative rather than quantitative. This has lead to the development of qualitative spatial reasoning (QSR) as a well-established subfield of artificial intelligence (AI). One of the most widely studied formal systems for QSR is the region-connection calculus (RCC) (Cohen et al. 1997). This system provides an axiomatization of space in which regions themselves are primitives, rather than being constructed from more primitive sets of points. Such an axiomatization is one way to remove the dependence of spatial descriptions on the real numbers, but RCC is still a theory in which regions are considered to be "crisp". That is, regions have ideal boundaries, and even for an arbitrarily small region, r, we can in principle determine whether r lies inside, outside or straddles the boundary of any other region. In spatial data obtained in the real world, perhaps by remote sensing via satellite, we cannot always expect to be able to distinguish these three cases with certainty.

The work on RCC provides a valuable foundation for QSR, but its basis in crisp regions is a limitation which has been acknowledged in the development of "egg-yolk" regions (Cohn and Gotts 1996; Roy and Stell 2001). Egg-yolk regions address issues of vagueness and indeterminacy (Burrough and Frank 1995) but are not intended to deal with levels of detail in descriptions. The fact that spatial data cannot be described with absolute accuracy shows that we need to deal with descriptions incorporating only a certain finite level of detail. This has also been addressed in the work of Worboys (1998a, 1998b). This is not however the only motivation for what we call approximate regions in this paper, that is regions described only up to some particular level of detail. Another reason for the significance of such regions lies in human cognitive abilities. With limitations on short-term memory (Miller 1956), information cannot be efficiently processed if unnecessary detail obscures the main features. This is especially so where the data is dynamic, as is the case when trying to navigate and relating the changing environment, as one moves, to a map, which might itself be dynamic. The importance of levels of detail in Marr's (1982) work on vision as information processing is also noteworthy here.

New technological means of delivering spatial data provide further motivation for considering level of detail. For example, the provision of maps over the Internet, and even more so to mobile phones, shows that excessive detail will either lead to unacceptable delivery times, or to data which is too complex to be usable by humans. The considerable body of work on cartographic generalization (Müller et al. 1995), that is the derivation of less detailed maps from more detailed ones, shows that level of detail has long been an important issue in traditional map-making. Traditionally generalization has been concerned with the relationship between maps at different scales. Generalization is a topic of importance all the more so now that the notion of scale for spatial data is no longer a clear-cut one (Goodchild and Proctor 1998).

Having seen that there are several reasons for the consideration of spatial data at multiple levels of detail, we have to consider how these various levels of approximation can be represented. One possibility is to retain the idea of a coordinate system, but in a qualitative way. The cartesian frame of reference is replaced by a set of regions, F, forming a framework with respect to which regions are described. follows the principal that we absolute sense, only how it relates ways of providing a qualitative description of a region R with respect to such a framework. These vary depending on what relationships between R and the cells in the framework are admitted. There are also options in whether we place restrictions on the framework. It might be required that the framework be both comprehensive and irredundant. That is, every region we want to describe has no parts lying outside all the elements of the framework, and no part of the region lies in two different members of the framework. These restrictions are frequently adopted, and we shall use them ourselves in the technical parts of the paper, but they are not essential to the idea of qualitative coordinatization.

The present paper builds on our earlier work (Bittner and Stell 1998), in which we detailed various ways of providing qualitative representations of regions with respect to a partition of the plane. Here, we develop the framework we established in that paper to deal with systems of relations between regions which are qualitatively approximated. Thus we extending the body of work on RCC concerned with relations between regions. For example two regions could be overlapping, or perhaps only touch at their boundaries. There are two principal schemes of relations between crisp RCC regions: five relations known as RCC5, and eight known as RCC8. In the present paper we demonstrate how the RCC5 and RCC8 schemes can be extended from ideal regions to approximate ones.

The cognitive adequacy of systems of relations between crisp regions has been investigated (Knauff et al. 1995; Knauff et al. 1997; Renz et al 2000), and of course the same needs to be done for relations on approximate regions. We hope that researchers will thus take up the challenge of the cognitive properties of approximate spatial regions and of relations between them. Formal work such as that reported here can be used as a basis for these cognitive investigations.

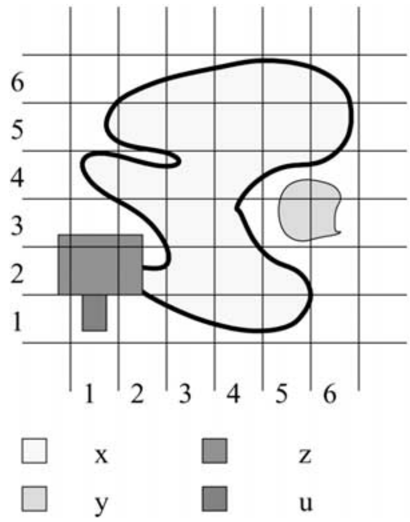
2. Approximating regions

Spatial regions can be described by specifying how they relate to a frame of reference. In the case of two-dimensional regions, the frame of reference could be a partition of the plane into cells which may share boundaries but which do not overlap. A region can then be described by giving the relationship between the region and each cell (Bittner and Stell 1998) introduced the notions of boundary sensitive and boundary insensitive approximations which will be reviewed in this section. Boundary sensitive approximations take the relationships between the region and boundary segments shared by neighboring partition cells into account.

2.1. *Spatial regions*

In this paper we assume that regions are regular and satisfy the axioms of the RCC-theory (Randall et al. 1992). This means that the boundary is "connected" to the region it bounds and that every region is identical to its closure (Gotts 1996). Consequently, regions can be modeled as regular closed sets. Regular closed sets form a complete boolean algebra with meet and join operations $\wedge$ and $\vee$ (Halmos 1963), which are interpreted as intersection and union operations on regions. In this paper we consider regular regions of two and one dimensional space that are embedded in the plane. It is important to notice that the meet of two regular regions is either empty, $x \wedge y = \perp$, or non-empty, i.e., $x \wedge y \neq \perp$. If $x \wedge y \neq \perp$ then the result is a regular region of the same dimension of x and y (Halmos 1963). Let x be a (regular and planar) region of dimension two then the boundary operator δx yields a regular region of dimension one. Let x and y be two-dimensional regions then we have $\delta x \wedge \delta y \neq \perp$ if and only if x and y share a boundary segment, i.e., a (regular) region of dimension one.

We use the notion $\cap$ in order to refer to intersection operations between (regular) regions of different dimension. Given the interpretation of regions



$$X_3 = \begin{array}{c||c|c|c|c|c|c} G & (1,6) & (2,6) & (3,6) & \dots & (3,5) & \dots \\ \hline \Omega_3 & \text{no} & \text{po} & \text{po} & \dots & \text{fo} & \dots \end{array}$$

$$Y_3 = \begin{array}{c||c|c|c|c|c|c} G & \dots & (5,4) & (6,4) & (5,3) & (6,3) & \dots \\ \hline \Omega_3 & \dots & \text{po} & \text{po} & \text{po} & \text{po} & \dots \end{array}$$

Figure 1. Rough approximations of spatial regions.

as regular sets then $\cap$ can be interpreted as the standard intersection of sets. The outcome of $\cap$ it not necessarily a regular set. Consequently we have $x \cap y \neq \perp$ if the two-dimensional regions x and y share a single boundary point but we have $x \wedge y = \perp$ and $\delta x \wedge \delta y = \perp$.

2.2. *Boundary insensitive approximation*

2.2.1. *Approximation functions*

Suppose a space R of detailed or precise regions. By imposing a partition, G, on R we can approximate elements of R by elements of Ω_3^G. That is, we approximate regions in R by functions from G to the set $\Omega_3 = \{$**fo, po, no**$\}$. The function which assigns to each region $r \in R$ its approximation will be denoted $\alpha_3 : R \to \Omega_3^G$. The value of $(\alpha_3 r)g$ is **fo** if r covers all the of the cell g, it is **po** if r covers some but not all of the interior of g, and it is **no** if there is no overlap between r and g. We call the elements of Ω_3^G the boundary insensitive approximations of regions $r \in R$ with respect to the underlying regional partition G. Consider Figure 1. Approximation mapping representations, $(\alpha_3\, x)$ and $(\alpha_3\, y)$ of the regions x and y are given. Partition cells are denoted by pairs of integer numbers referring to columns and rows. Notice that we use a raster-shaped partition since it is easier to draw and to refer to specific partition cells of the underlying raster-shaped partition. In general the partition can consist of arbitrarily shaped regions. In the remainder of this paper we use capital letters in order to refer to approximations, i.e., X_3 instead of $(\alpha_3\, x)$. Wherever the context is clear the subscript is omitted.

2.2.2. *Semantics of approximate regions*

Each approximate region $X \in \Omega_3^G$ stands for a set of precise regions, i.e., all those precise regions having the approximation X. This set which will be denoted $\|X\|_3$ provides a semantics for approximate regions.

$$\|X\|_3 = \{r \in R \mid \alpha_3 r = X\}$$

Where ever the context is clear we omit the superscript.

2.2.3. *Operations on approximation functions*

The domain of regions is equipped with a meet operation interpreted as the intersection of regions. In the domain of approximation functions the meet operation between regions is approximated by pairs of greatest minimal, $\underline{\wedge}$, and least maximal, $\overline{\wedge}$, meet operations on approximation mappings (Bittner and Stell 1998).

Consider the operations $\underline{\wedge}$ and $\overline{\wedge}$ on the set $\Omega_3 = \{$**fo, po, no**$\}$ that are defined as follows.

$\underline{\wedge}$	no	po	fo
no	no	no	no
po	no	no	po
fo	no	po	fo

$\overline{\wedge}$	no	po	fo
no	no	no	no
po	no	po	po
fo	no	po	fo

These operations extend to elements of Ω_3^G (i.e., the set of functions from G to Ω_3) by

$$(X \,\underline{\wedge}\, Y)g \;=\; (Xg) \,\underline{\wedge}\, (Yg)$$

and similarly for $\overline{\wedge}$. This definition of the operations on Ω_3^G is equivalent to the construction for operations given by Bittner and Stell (1998, p. 108).

An example using approximations the regions x and y in Figure 1 is given in the table below. The operation $X \,\underline{\wedge}\, Y$ yields $\perp$, i.e., the function mapping all elements of G onto **no**, and the operation $X \,\overline{\wedge}\, Y$ yields a value different from $\perp$. This reflects the fact that there are regions in $x' \in \|X\|$ and $y' \in \|Y\|$ that do have a non-empty meet, i.e., $x' \wedge y' \neq \perp$, and there are regions in

$x'' \in \|X\|$ and $y'' \in \|Y\|$ that do have an empty meet, i.e., i.e., $x'' \wedge y'' = \bot$ (for example the regions x and y depicted in Figure 1).

G	...	(5,4)	(6,4)	...	(5,3)	(6,3)	...
$X = (\alpha\ x)$	...	po	po	...	no	no	...
$Y = (\alpha\ y)$	...	po	po	...	po	po	...
$X \underline{\wedge} Y$	...	no	no	...	no	no	...
$X \overline{\wedge} Y$	...	po	po	...	no	no	...

2.3. *Boundary sensitive approximation*

2.3.1. *Approximation functions*

We can refine the approximation of regions R with respect to the partition G by taking boundary segments shared by neighboring partition regions into account. That is, we approximate regions in R by functions from $G \times G$ to the set $\Omega_5 = \{$**fo, fbo, pbo, nbo, no**$\}$. The function which assigns to each region $r \in R$ its boundary sensitive approximation will be denoted $\alpha_5 : R \to \Omega_5^{G \times G}$. The value of $(\alpha_5 r)(g_i, g_j)$ is **fo** if r covers all of the cell g_i, it is **fbo** if r covers all of the boundary segment, (g_i, g_j), shared by the cell g_i and g_j and some but not all of the interior of g_i, it is **pbo** if r covers some but not all of the boundary segment (g_i, g_j) and some but not all of the interior of g_i, it is **nbo** if r does not intersect with boundary segment (g_i, g_j) and some but not all of the interior of g_i, and it is **no** if there is no overlap between r and g_i.

Let bs be the boundary segment shared by the cells g_i and g_j, i.e., $\delta g_i \wedge \delta g_j \neq \bot$. Approximation mappings, α_3, apply to configurations of regions in one and two-dimensional space. We define boundary sensitive approximation, α_5, in terms of pairs of approximation mappings, α_3, according to the intuitive definition above. The operation α_3 in $(\alpha_3\ r)g_i$ operates on the two-dimensional regions g_i and r. The operation α_3 in $(\alpha_3\ (r \cap bs))bs$ operates on one-dimensional regions $r \cap (\delta g_i \wedge \delta g_j)$ and $(\delta g_i \wedge \delta g_j)$. We define the values of $(\alpha_5\, r)(g_i, g_j)$ as in the following table. The value depends on $(\alpha_3\ r)g_i$ and on $(\alpha_3\ (r \cap bs))bs$.

	$(\alpha_3\ (r \cap bs))bs =$		
	fo	po	no
$(\alpha_3\ r)g_i = \text{fo}$	fo	–	–
$(\alpha_3\ r)g_i = \text{po}$	fbo	pbo	nbo
$(\alpha_3\ r)g_i = \text{no}$	no	no	no

The pairs with $((\alpha_3\, r)\, g_i) =$ **fo** and $((\alpha_3\, (r \cap bs))\, bs) \neq$ **fo** cannot occur since $((\alpha_3\, r)\, g_i) =$ **fo** means that r covers all of g_i including its boundary.

If $((\alpha_3\, r)\, g_i)$ = **no** then the result of $((\alpha_3\ (r \cap bs))\, bs)$ does not matter since for $(\alpha_5\ r)(g_i, g_j) \neq$ **no** the region r and the cell g_i must overlap, i.e., share interior parts. The values **fo, fbo, pbo, nbo, no** are abbreviations for pairs $(\omega_\iota, \omega_\delta) \in \Omega_3 \times \Omega_3$. An example of the boundary sensitive approximation of the regions u and z in Figure 1 is given below.

$U_5 =$

$G \times G$	((1, 1), (1, 2))	((1, 1), (2, 1))	((2, 1), (1, 1))	((2, 2), (2, 1))	...
Ω_5	pbo	nbo	no	no	...

$Z_5 =$

$G \times G$	((1, 1), (1, 2))	((1, 1), (2, 1))	((2, 1), (1, 1))	((2, 2), (2, 1))	...
Ω_5	no	no	fo	pbo	...

Let ω be an element of the boundary sensitive value domain Ω_5 with $\omega = (\omega_\iota, \omega_\delta)$. We call $\omega_\iota = (\iota(\omega_\iota, \omega_\delta))$ the interior component and $\omega_\delta = (\delta\ (\omega_\iota, \omega_\delta))$ the boundary component of ω.

Each approximate region $X \in \Omega_5^{G\times G}$ stands for a set of precise regions, i.e., all those precise regions having the approximation X. This set which will be denoted $\|X\|_5$ provides a semantics for approximate regions. $\|X\|_5 = \{r \in R \mid \alpha_5 r = X\}$ Where ever the context is clear we omit the superscript.

2.3.2. *Operations on boundary sensitive approximations*

We define the operation $\overline{\wedge}$ on the set Ω_5 = {**fo, fbo, pbo, nbo, no**} as:

$\overline{\wedge}$	no	nbo	pbo	fbo	fo
no	no	no	no	no	no
nbo	no	nbo	nbo	nbo	nbo
pbo	no	nbo	pbo	pbo	pbo
fbo	no	nbo	pbo	fbo	fbo
fo	no	nbo	pbo	fbo	fo

These operations extend to elements of $\Omega_5^{G\times G}$ (i.e., the set of functions from $G \times G$ to Ω_5) by $(X \mathbin{\overline{\wedge}} Y)(g_i, g_j) = (X(g_i, g_j)) \mathbin{\overline{\wedge}} (Y(g_i, g_j))$. An example using the approximations of the regions x and z in Figure 1 is given below.

The definition of the operation $\underline{\wedge}$ is more complicated. It is not local, i.e., there are cases there not only the boundary segment (g_i, g_j) needs to be taken into account, but all boundary segments of the cell g_i. For the purpose of this paper it is sufficient to define:

$\underline{\wedge}$	no	nbo	pbo	fbo	fo
no	no	no	no	no	no
nbo	no	no$\mid \leq$ nbo	no$\mid \leq$ nbo	no$\mid \leq$ nbo	nbo
pbo	no	no$\mid \leq$ nbo	no$\mid \leq$ nbo	pbo	pbo
fbo	no	no$\mid \leq$ nbo	pbo	fbo	fbo
fo	no	nbo	pbo	fbo	fo

In cases where two values are given the left value refers to the outcome of $(X(g_i, g_j)) \underline{\triangle} (Y(g_i, g_j))$ (the local definition). The constraint given on the right refers to the possible outcome of the (non-local) operation

$(X(g_i, g_j)) \underline{\triangle}^{N(g_i)} (Y(g_i, g_j))$, which takes all boundary segments of the cell g_i into account. If only one value is given then local and non-local operations yield the same result. For details see (Bittner and Stell 1998). An example using the approximations of the regions x and z in Figure 1 is given below.

$G \times G$	$((1,2),(1,1))$	$((2,2),(1,2))$	$((2,2),(2,1))$	$((2,2),(3,2))$	...
X_5	no	nbo	pbo	pbo	...
Z_5	fo	fbo	pbo	nbo	...
$X_5 \underline{\triangle} Z_5$	no	no	no	no	...
$X_5 \overline{\wedge} Z_5$	no	nbo	pbo	nbo	...

3. Redefining RCC relations

In this section we propose a specific style of defining RCC relations. This style allows to define RCC relations exclusively based on constraints regarding the outcome of the meet operation between (one and two dimensional) regions. Furthermore this style of definitions allows us to obtain a partial ordering with minimal and maximal element on the relations defined. Both aspects are critical for the generalization of these relations to the approximation case.

3.1. *RCC5 relations*

Given two regions x and y the RCC5 relation between them can be determined by considering the triple of boolean values:

$$(x \wedge y \neq \bot, \; x \wedge y = x, \; x \wedge y = y).$$

The correspondence between such triples and the RCC5 classification is given in Table 1. Possible geometric interpretations are given in Figure 2.

The set of triples is partially ordered by setting $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ iff $a_i \leq b_i$ for $i = 1, 2, 3$, where the Boolean values are ordered by $\mathbf{F} < \mathbf{T}$. This is the same ordering induced by the RCC5 conceptual graph (Goodday and Cohn 1994). But note that the conceptual graph has **PO** and **EQ** as neighbors which is not the case in the Hasse diagram for the partially ordered

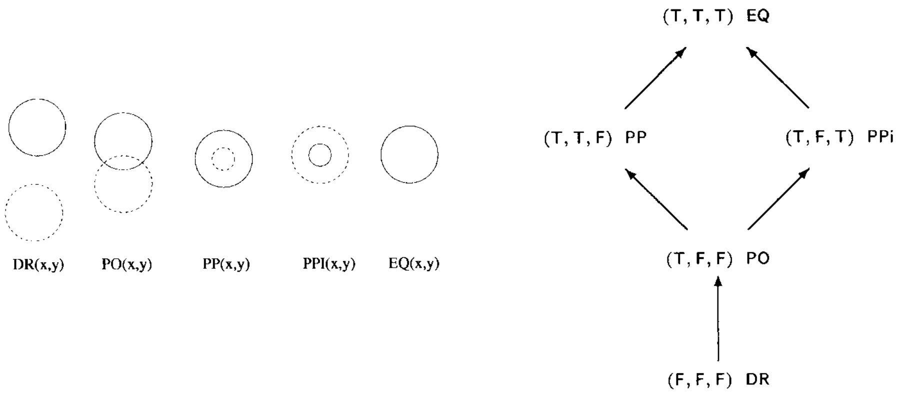


Figure 2. RCC5 relations and RCC5 lattice.

Table 1. Definition of theRCC8 relations

$x \wedge y \neq \bot$	$x \wedge y = x$	$x \wedge y = y$	RCC5
F	F	F	DR
T	F	F	PO
T	T	F	PP
T	F	T	PPi
T	T	T	EQ

set (The right diagram in Figure 2). We refer to this as the RCC5 *lattice* to distinguish it from the conceptual neighborhood graph.

3.2. *RCC8 relations*

In order to describe RCC8 relations we define the relationship between x and y by using a triple, but where the three entries may take one of three truth values rather than the two Boolean ones. The scheme has the form

$$(x \wedge y \not\approx \bot, x \wedge y \approx x, x \wedge y \approx y)$$

where

$$x \wedge y \not\approx \bot = \begin{cases} \mathbf{T} & \text{if the interiors of } x \text{ and } y \text{ overlap, i.e., } x \wedge y \neq \bot \\ \mathbf{M} & \text{if only the boundaries } x \text{ and } y \text{ overlap, i.e.,} \\ & \quad x \wedge y = \bot \text{ and } \delta x \wedge \delta y \neq \bot \\ \mathbf{F} & \text{if there is no overlap between } x \text{ and } y \text{, i.e.,} \\ & \quad x \wedge y = \bot \text{ and } \delta x \wedge \delta y = \bot \end{cases}$$

Table 2. The definition table of the RCC8 relations

$x \wedge y \not\approx \perp$	$x \wedge y \approx x$	$x \wedge y \approx y$	RCC8
F	F	F	DC
M	F	F	EC
T	F	F	PO
T	M	F	TPP
T	T	F	NTPP
T	F	M	TPPi
T	F	T	NTPPi
T	T	T	EQ

and where[1]

$$x \wedge y \approx x = \begin{cases} \mathbf{T} & \text{if either } x = y \text{ or } x \text{ is contained in the interior of } y, \\ & \text{i.e., } x \wedge y = x \text{ and } \delta x \wedge \delta y = \perp \\ \mathbf{M} & \text{if } x \text{ is contained in } y \text{ and the boundaries overlap, i.e.,} \\ & x \wedge y = x \text{ and } x \wedge y \neq y \text{ and } \delta x \wedge \delta y \neq \perp \\ \mathbf{F} & \text{if } x \text{ is not contained within } y, i.e., x \wedge y \neq x \end{cases}$$

and similarly for $x \wedge y \approx y$. The meaning of $x \wedge y \neq \perp$ is that the intersection of the interior of x and y is non-empty and the meaning of $\delta x \wedge \delta y = \perp$ is that the meet of the boundaries of x and y is empty.[2] The correspondence between triples $(x \wedge y \not\approx \perp, x \wedge y \approx x, x \wedge y \approx y)$ and the RCC8 classification is given in Table 2.

Consider the definition of the relation $\mathbf{DC}(x, y)$. By Table 2 we have $x \wedge y \not\approx \perp = \mathbf{F}$, $x \wedge y \approx x = \mathbf{F}$, and $x \wedge y \approx y = \mathbf{F}$. Consequently, neither the interiors nor the boundaries of x and y overlap, i.e., $x \wedge y = \perp$ and $\delta x \wedge \delta y = \perp$, and the regions x and y are disconnected. In the case of $\mathbf{EC}(x, y)$ we have $x \wedge y \not\approx \perp = \mathbf{M}$, $x \wedge y \approx x = \mathbf{F}$, and $x \wedge y \approx y = \mathbf{F}$. Consequently, the interiors of x and y do not overlap but the boundaries do, i.e., $x \wedge y = \perp$ and $\delta x \wedge \delta y \neq \perp$, and the regions x and y are externally connected. In the case of $\mathbf{NTPP}(x, y)$ we have $x \wedge y \not\approx \perp = \mathbf{T}$, $x \wedge y \approx x = \mathbf{T}$ and $x \wedge y \approx y = \mathbf{F}$. Consequently, x is completely contained in the interior of y, i.e., $x \wedge y \neq \perp$, $x \wedge y = x$ and since $x \wedge y \neq y$ we have $\delta x \wedge \delta y = \perp$, i.e., x is a non-tangential proper part of y. In the case of $\mathbf{EQ}(x, y)$ we have $x \wedge y \not\approx \perp = \mathbf{T}$, $x \wedge y \approx x = \mathbf{T}$ and $x \wedge y \approx y = \mathbf{T}$. Both regions are identical, i.e., $x \wedge y = x$, $x \wedge y = y$, and $\delta x \wedge \delta y = \delta x = \delta y$.

The RCC5 relation **DR** *refines* to **DC** and **EC**, the RCC5 relation **PP** *refines* to **TPP** and **NTPP**, and the RCC5 relation **PPi** *refines* to **TPPi** and **NTPPi**. The set of triples is partially ordered as discussed above and the truth

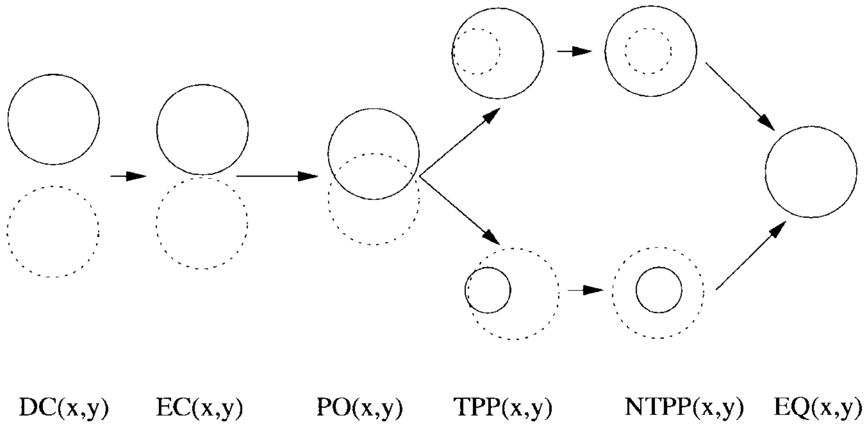


Figure 3. RCC8 relations and lattice.

values are ordered by **F** < **M** < **T**. We call the corresponding Hasse diagram (Figure 3) RCC8 *lattice* to distinguish it from the conceptual neighborhood graph (Goodday and Cohen 1994).

4. Generalizations of RCC5 relations

The original formulation of RCC dealt with ideal regions which did not suffer from imperfections such as vagueness, indeterminacy or limited resolution. However, these are factors which affect spatial data in practical examples, and which are significant in applications such as geographic information systems (GIS), e.g. (Burrough and Frank 1995). The issue of vagueness and indeterminacy has been tackled in the work of (Cohn and Gotts 1996). The topic of the present paper is not vagueness or indeterminacy in the widest sense, but rather the special case where spatial data is approximated by being given a limited resolution description. The question we are going to address in the remainder of this paper is: Given a limited resolution of description what can we say about relations that can hold between the approximated regions? In order to do so we propose to generalize the topological relations between regions discussed in the previous section in such a way that they apply to boundary insensitive and to boundary sensitive approximations.

4.1. *Semantic and syntactic generalizations*

There are two approaches we can take to generalizing the RCC5 classification from precise regions to approximate ones. These two may be called the semantic and the syntactic.

Semantic We can define the RCC5 relationship between approximate regions X and Y to be the set of relationships which occur between any pair of precise regions representing X and Y. That is, we can define

$$SEM(X, Y) = \{RCC5(x, y) \mid x \in \|X\| \text{ and } y \in \|Y\|\}.$$

Syntactic We can take a formal definition of RCC5 in the precise case which uses operations on R and generalize this to work with approximate regions by replacing the operations on R by analogous ones for Ω^G.

The syntactic generalization has many variants since there are many different ways in which the RCC5 can be formally defined in the precise case, and some of these can be generalized in different ways to the approximate case. The fact that several different generalizations can arise from the same formula is because some of the operations in R (such as $\wedge$ and $\vee$) have themselves more than one generalization to operations on Ω^G. It is important to note that there is no *a priori* reason to suppose that any two of these various generalizations (either the semantic and one syntactic one, or two syntactic ones) will be equivalent to each other in any sense. Relationships between the various possibilities have to be investigated and any equivalences need to be stated precisely and proved.

4.2. *Syntactic generalization*

The above formulation of the RCC5 relations can be extended to approximate regions. One way to do this is to perform the following three steps: Assuming definitions of relations between regions exclusively based on the meet operation we, firstly, replace in the definitions of the RCC5 relations the variables ranging over regions by variables ranging over approximations and secondly we replace the meet operation, $\wedge$, between regions by the greatest minimal operation, $\underline{\wedge}$ between approximations, and thirdly we replace $\wedge$ by the least maximal operation, $\overline{\wedge}$. If X and Y are approximate regions (i.e. functions from G to Ω_3) then we can consider the two triples of Boolean values:

$$\begin{aligned}
&(X \underline{\wedge} Y \neq \bot, X \underline{\wedge} Y = X, X \underline{\wedge} Y = Y),\\
&(X \overline{\wedge} Y \neq \bot, X \overline{\wedge} Y = X, X \overline{\wedge} Y = Y).
\end{aligned} \tag{1}$$

In the context of approximate regions, the bottom element, $\bot$, is the function from G to Ω_3 which takes the value no for every element of G. Each of the above triples provides an RCC5 relation, so the relation between X and Y can be measured by a pair of RCC5 relations. These relations will be denoted by $\underline{R}(X, Y)$ and $\overline{R}(X, Y)$.

THEOREM 1. *The pairs* $(\underline{R}(X, Y), \overline{R}(X, Y))$ *which can occur are all pairs* (a, b) *where* $a \leq b$ *with the exception of (***PP, EQ***) and (***PPi, EQ***).*

Proof First we show that $\underline{R}(X, Y) \leq \overline{R}(X, Y)$. Suppose that $\underline{R}(X, Y) = (a_1, a_2, a_3)$ and that $\overline{R}(X, Y) = (b_1, b_2, b_3)$. We have to show that $a_i \leq b_i$ for $i = 1, 2, 3$. Taking the first component, if $X \underline{\wedge} Y \neq \perp$ then for each g such that $Xg \underline{\wedge} Yg \neq$ **no**, we also have, by examining the tables for $\underline{\wedge}$ and $\overline{\wedge}$, that $Xg \overline{\wedge} Yg \neq$ **no**. Hence $X \overline{\wedge} Y \neq \perp$. Taking the second component, if $X \underline{\wedge} Y = X$ then $X \overline{\wedge} Y = X$ because from $Xg \underline{\wedge} Yg = Xg$ it follows that $Xg \overline{\wedge} Yg = Xg$. This can be seen from the tables for $\underline{\wedge}$ and $\overline{\wedge}$ by considering each of the three possible values for Xg. The case of the third component follows from the second since $\underline{\wedge}$ and $\overline{\wedge}$ are commutative.

Finally we have to show that the pairs (**PP, EQ**) and (**PPi, EQ**) cannot occur. If $\overline{R}(X, Y) =$ **EQ**, then $X = Y$ so $X \underline{\wedge} Y = X$ must take the same value as $X \underline{\wedge} Y = Y$. Thus the only triples which are possible for $\underline{R}(X, Y)$ are those where the second and third components are equal. This rules out the possibility that $\underline{R}(X, Y)$ is **PP** or **PPi**.

4.3. *Correspondence of semantic and syntactic generalization*

Let the syntactic generalization of RCC5 defined by

$$SYN(X, Y) = (\underline{R}(X, Y), \overline{R}(X, Y)),$$

where $\underline{R}$ and $\overline{R}$ are as defined above.

THEOREM 2. *For any approximate regions X and Y, the two ways of measuring the relationship of X to Y are equivalent in the sense that*

$$SEM(X, Y) = \{\rho \in RCC5 \mid \underline{R}(X, Y) \leq \rho \leq \overline{R}(X, Y)\},$$

where RCC5 is the set {**EQ, PP, PPi, PO, DR**}, *and* $\leq$ *is the ordering in the RCC5 lattice.*

The proof of this theorem depends on assumptions about the set of precise regions R. We assume that R is a model of the RCC axioms so that we are approximating continuous space, and not approximating a space of already approximated regions.

Proof There are three things to demonstrate. Firstly that for all $x \in \|X\|$, and $y \in \|Y\|$, that $\underline{R}(X, Y) \leq RCC5(x, y)$. Secondly, for all x and y as before, that $RCC5(x, y) \leq \overline{R}(X, Y)$, and thirdly that if ρ is any RCC5 relation such that $\underline{R}(X, Y) \leq \rho \leq \overline{R}(X, Y)$ then there exist particular x and y which stand in the relation ρ to each other. To prove the first of these it is necessary to consider each of the three components $X \underline{\wedge} Y \neq \perp$, $X \underline{\wedge} Y = X$ and $X \underline{\wedge} Y = Y$ in turn. If $X \underline{\wedge} Y \neq \perp$ is true, we have to show for all $x \in \|X\|$ and $y \in \|Y\|$ that $x \wedge y \neq \perp$ is also true. From $X \underline{\wedge} Y \neq \perp$ it follows that there is at least one cell g where one of X and Y fully overlaps g, and the other at least partially overlaps g. Hence there are interpretations of X and Y having non-empty intersection. If $X \underline{\wedge} Y = X$ is true then for all cells g we have $Xg = \mathbf{no}$ or $Yg = \mathbf{fo}$. In each case every interpretation must satisfy $x \wedge y = x$. Note that this depends on the fact that the combination $Xg = \mathbf{po} = Yg$ cannot occur. The case of the final component $X \underline{\wedge} Y = Y$ is similar. Thus we have demonstrated for all $x \in \|X\|$ and $y \in \|Y\|$ that $\underline{R}(X, Y) \leq RCC5(x, y)$. The task of showing that $RCC5(x, y) \leq \overline{R}(X, Y)$ is accomplished by a similar analysis. Finally, we have to show that for each RCC5 relation, ρ, where $\underline{R}(X, Y) \leq \rho \leq \overline{R}(X, Y)$, there are $x \in \|X\|$ and $y \in \|Y\|$ such that the relation of x to y is ρ. This is done by considering the various possibilities for $\underline{R}(X, Y)$ and $\overline{R}(X, Y)$. We will only consider one of the cases here, but the others are similar. If $\underline{R}(X, Y) = \mathbf{PO}$ and $\overline{R}(X, Y) = \mathbf{EQ}$, then for each cell g, the values of Xg and Yg are equal and there must be some cells where this value is **po** and some cells where the value is **fo**. Precise regions $x \in \|X\|$ and $y \in \|Y\|$ can be constructed by selecting sub-regions of each cell g say x_g and y_g, and defining x and y to be the unions of these sets of sub-regions. In this particular case, there is sufficient freedom with those cells where $Xg = Yg = \mathbf{po}$ to be able to select x_g and y_g so that the relation of x to y can be any ρ where $\mathbf{PO} \leq \rho \leq \mathbf{EQ}$.

5. Generalizations of RCC8 relations

In this section we discuss the generalization of RCC8 relations. Essentially we apply the same techniques we discussed in the previous section. In order to do so we first discuss how to convert expressions like $\delta(X) \underline{\wedge} \delta(Y) \neq \bot$ to the domain of boundary sensitive approximation. Then we discuss the syntactic and semantic generalization of RCC8 relations.

5.1. *Intersection at boundary segments*

When we refined the RCC5 relations to RCC8 relations in Section 3 we took the outcome of the operation $\delta x \wedge \delta y$ into account. In order to generalize the definitions of RCC8-relations to boundary sensitive approximations in a similar way we generalized RCC5-relations we need to define formulas $\delta(X) \underline{\wedge} \delta(Y) \neq \bot$ and $\delta(X) \overline{\wedge} \delta(Y) \neq \bot$ corresponding to the formula $\delta(x) \wedge \delta(y) \neq \bot$ that was used in the definitions of the RCC8 relations. Assuming the partial order of the RCC8-lattice we want $\delta(X) \underline{\wedge} \delta(Y) \neq \bot$ to be true if and only if the least RCC8-relation that can hold between $x \in \|X\|$ and $y \in \|Y\|$ involves boundary intersection at a boundary segment of G.[3] We want $\delta(X) \overline{\wedge} \delta(Y) \neq \bot$ to be true if and only if the greatest RCC8-relation

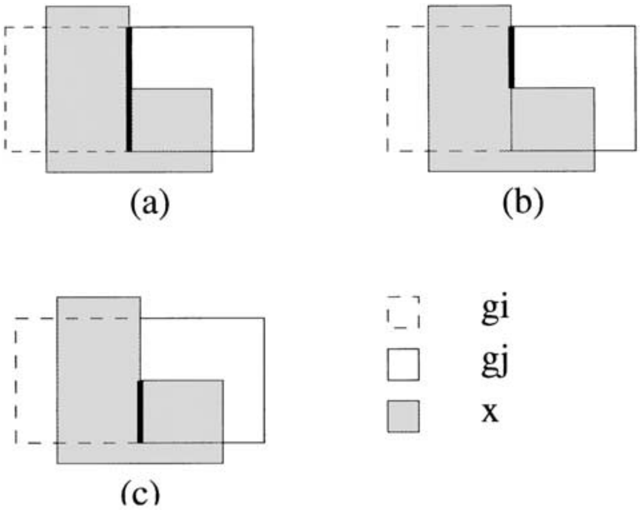


Figure 4. Intersection of x as a whole (a), the boundary of x (b), and the interior of x (c), with the boundary segment shared by the cells g_i and g_j.

that can hold between regions $x \in \|X\|$ and $y \in \|Y\|$ involves boundary intersection at a boundary segment in G. In the remainder we use the notion of a pair $(g_i, g_j) \in G \times G$ in order to the boundary segment of the partition region g_i shared with the neighboring partition region g_j.[4]

5.1.1. *Approximate intersections at boundary segments*

Consider Figure 4. The regions g_i and g_j share the boundary segment (g_i, g_j). Given a third region, x, we can ask which parts of x intersect with the boundary segment (g_i, g_j). Given a one-dimensional intersection of x and the boundary segment (g_i, g_j) and the distinction between interior and boundary parts of x, we can identify three (possibly empty) subsets of $((\delta g_i \wedge \delta g_j) \cap x)$: (a) the intersection of the of x as a whole (i.e., interior and boundary) with $(\delta g_i \wedge \delta g_j)$, $x \cap (\delta g_i \wedge \delta g_j)$; (b) the intersection of boundary parts of x with $(\delta g_i \wedge \delta g_j))$, $\delta(x) \wedge (\delta g_i \wedge \delta g_j)$; (c) the intersection of interior parts of x with $((\delta g_i \wedge \delta g_j)))$, $\iota(x) \cap (\delta g_i \wedge \delta g_j))$. The bold lines of the in Figure 4 mark the corresponding parts of the boundary segment (g_i, g_j).

Given the boundary sensitive approximation X we can easily decide for each boundary segment, (g_i, g_j), whether parts of the interior, parts of the boundary, or parts of both, the interior and the boundary, of regions $x \in \|X\|$ intersect this boundary segment. Furthermore, we can derive the degree of coverage, (**fo** or **po** or **no**), of (g_i, g_j) by the interior/boundary/whole of x: Let $\delta(X(g_i, g_j))$ be the boundary component[5] of $(X(g_i, g_j))$ and let $\delta(X(g_j, g_i))$ be the boundary component of $(X(g_j, g_i))$ respectively. We define $\pi(X, (g_i, g_j))$ to be the approximation of the intersection of $x \in \|X\|$ and the boundary segment (g_i, g_j):[6]

$$\pi(X, (g_i, g_j)) = \max(\delta(X(g_i, g_j)), \delta(X(g_j, g_i))).$$

Due to the definition of X we have $\pi(X, (g_i, g_j))$ = **fo** if x covers (g_i, g_j), $\pi(X, (g_i, g_j))$ = **po** if x covers parts of (g_i, g_j), and $\pi(X, (g_i, g_j))$ = **no** if x does not overlap (g_i, g_j). Consider configuration (a) in Figure 4: We have $(X(g_i, g_j))$ = (**po, fo**), $(X(g_j, g_i))$ = (**po, po**), $\pi(X, (g_i, g_j))$ = max(**fo, po**) = **fo**.

In order to define the approximation of the intersection of the interior and the boundary of x and the boundary segment (g_i, g_j) we need to define an operation $\ominus : \Omega_3 \times \Omega_3 \to \Omega_3$ in analogy to the subtraction of regions:[7]

$\ominus$	no	po	fo
no	no	po	fo
po	po	no	po
fo	fo	po	no

We now define the approximation of the intersection of the boundary and the interior of the regions x with respect to the boundary segment (g_i, g_j):

$$\pi^\delta(X, (g_i, g_j)) = \delta(X\,(g_i, g_j)) \ominus \delta(X\,(g_j, g_i))$$
$$\pi^\iota(X, (g_i, g_j)) = \pi(X, (g_i, g_j)) \ominus \pi^\delta(X, (g_i, g_j))$$

Consider configurations (b) and (c) in Figure 4: We have $(X(g_i, g_j))$ = (**po, fo**), $(X(g_j, g_i))$ = (**po, po**), $\pi^\delta(X(g_i, g_j))$ = **fo** $\ominus$ **po** = *po*, $\pi^\iota(X(g_i, g_j)) = \pi(X(g_i, g_j)) \ominus \pi^\delta(X(g_i, g_j))$ = **po**.

Let X be a boundary sensitive approximation and let $p : R \times R \to \Omega_3$ be a function mapping pairs of (one-dimensional) regions onto Ω_3. Using the definition of boundary sensitive approximations and the definitions of **no, nbo, pbo, fbo**, and **fo** one can verify that:

LEMMA 3. *For all* $x \in \|X\|$:

- $\pi(X, (g_i, g_j)) = p((\delta g_i \wedge \delta g_j), (x \cap (\delta g_i \wedge \delta g_j))$
- $\pi^\delta(X, (g_i, g_j)) = p((\delta g_i \wedge \delta g_j), (\delta(x) \wedge (\delta g_i \wedge \delta g_j))$
- $\pi^\iota(X, (g_i, g_j)) = p((\delta g_i \wedge \delta g_j), (\iota(x) \cap (\delta g_i \wedge \delta g_j)).$

5.1.2. *Operations*

Consider the greatest minimal and least maximal meet operations $\underline{\wedge}, \overline{\wedge}$: $\Omega_3 \times \Omega_3 \to \Omega_3$. These operations are defined for approximations of two-dimensional regions with respect to a partition of two-dimensional space as well as for approximations of one-dimensional regions with respect to a partition of one-dimensional space [1]. Consequently expressions like $\pi^{\delta}(X(g_i, g_j))\underline{\wedge}\pi^{\delta}(Y(g_i, g_j))$ and $\pi^{\delta}(X(g_i, g_j))\overline{\wedge}\pi^{\iota}(Y(g_i, g_j))$ are well defined and correspond to their two-dimensional counterparts.

5.1.3. *Deriving boundary intersections from boundary sensitive approximations*

We now define $\delta(X)\underline{\wedge}\delta(Y) \neq \perp$ and $\delta(X)\overline{\wedge}\delta(Y) \neq \perp$ formally. In order to derive $\delta(x) \wedge \delta(y) \neq \perp$ from approximations X and Y we need to take the approximation of the intersection of interior AND boundary parts of $x \in \|X\|$ and $y \in \|Y\|$ with boundary segments (g_i, g_j) into account. We need to distinguish two cases:

No interior parts of x and y intersect (g_i, g_j), i.e.,

$$\pi(X(g_i, g_j)) = \pi^{\delta}(X(g_i, g_j)) \text{ and } \pi(Y(g_i, g_j)) = \pi^{\delta}(Y(g_i, g_j)) \quad (2)$$

(See, for example, configurations a and b in Figure 5).

There are interior parts of x or y that intersect (g_i, g_j), i.e.,

$$\pi(X(g_i, g_j)) > \pi^{\delta}(X(g_i, g_j)) \text{ or } \pi(Y(g_i, g_j)) > \pi^{\delta}(Y(g_i, g_j)) \quad (3)$$

(See, for example, configurations c–f in Figure 5).

These distinctions are exhaustive since $\pi^{\delta} \leq \pi$ always holds. In the remainder we refer to these cases as case 2 and case 3.

At the formal level we need separate definitions for $\delta(X)\underline{\wedge}\delta(Y) \neq \perp$ for each of the two cases. For case 2 we define:

$$\begin{aligned}
&\delta(X)(\underline{\wedge}^2)\delta(Y) \neq \perp \equiv \\
&\quad \exists(g_i, g_j) : \pi^{\delta}(X(g_i, g_j))\underline{\wedge}\pi^{\delta}(Y(g_i, g_j)) \neq \perp \\
&\delta(X)(\overline{\wedge}^2)\delta(Y) \neq \perp \equiv \\
&\quad \exists(g_i, g_j) : \pi^{\delta}(X(g_i, g_j))\overline{\wedge}\pi^{\delta}(Y(g_i, g_j)) \neq \perp
\end{aligned} \quad (4)$$

Consider Figure 5. An example for $\delta(X)(\underline{\wedge}^2)\delta(Y) \neq \perp = \mathbf{T}$ is given in configuration a and an example for $\delta(X)(\overline{\wedge}^2)\delta(Y) \neq \perp = \mathbf{T}$ and $\delta(X)(\underline{\wedge}^2)\delta(Y) \neq \perp = \mathbf{F}$ is given in configuration b. For case 3 we define:

$$\begin{aligned}
&\delta(X)(\underline{\wedge}^3)\delta(Y) \neq \perp \equiv \\
&\quad \exists(g_i, g_j) : \pi^{\delta}(X(g_i, g_j))\overline{\wedge}\pi^{\delta}(Y(g_i, g_j)) \neq \perp \\
&\delta(X)(\overline{\wedge}^3)\delta(Y) \neq \perp \equiv \\
&\quad \exists(g_i, g_j) : \pi^{\delta}(X(g_i, g_j))\underline{\wedge}\pi^{\delta}(Y(g_i, g_j)) \neq \perp
\end{aligned} \quad (5)$$

In Figure 5 examples for $\delta(X)(\underline{\wedge}^3)\delta(Y) \neq \perp = \mathbf{T}$ and $\delta(X)(\underline{\wedge}^3)\delta(Y) \neq \perp = \mathbf{F}$ are given in configurations c and d. Configuration e provides an example for $\delta(X)(\underline{\wedge}^3)\delta(Y) \neq \perp = \mathbf{T}$.

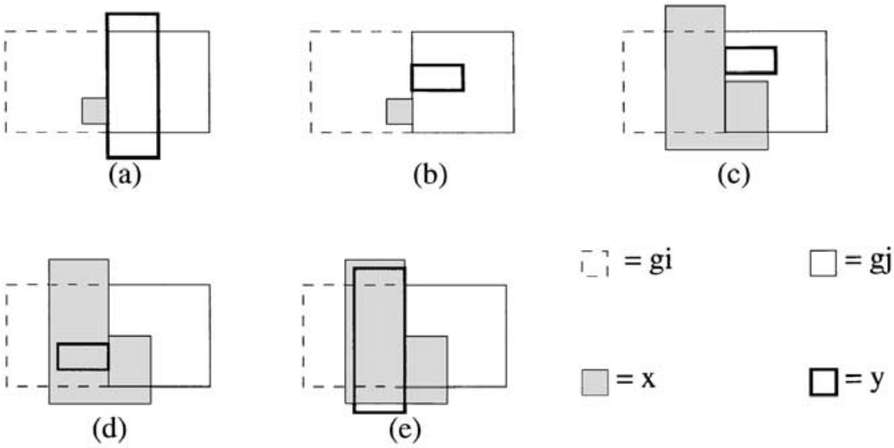


Figure 5. Intersection of $x \in \|X\|$ and $y \in \|Y\|$ at boundary segment (g_i, g_j).

In general we define:

$$\delta(X)\,\underline{\wedge}\,\delta(Y) \neq \bot \equiv\; \delta(X)(\underline{\wedge}^2\delta(Y) \neq \bot \;\text{ or }\delta(X)(\underline{\wedge}^3\delta(Y) \neq \bot$$
$$\delta(X)\,\overline{\wedge}\,\delta(Y) \neq \bot \equiv\; \delta(X)(\overline{\wedge}^2\delta(Y) \neq \bot \;\text{ or }\delta(X)(\overline{\wedge}^3\delta(Y) \neq \bot$$

These definitions ensure that $\delta(X)\,\underline{\wedge}\,\delta(Y) \neq \bot$ is true if and only if the least relation RCC8-relation that can hold between $x \in \|X\|$ and $y \in \|Y\|$ involves boundary intersection at a boundary segment in G and that $\delta(X)\,\overline{\wedge}\,\delta(Y) \neq \bot$ is true if and only if the greatest relation RCC8-relation that can hold between $x \in \|X\|$ and $y \in \|Y\|$ involves boundary intersection at a boundary segment in G. The formal proofs will be given in the Lemmata 7 and 8 below.

5.2. *Syntactic generalization of RCC8 relations*

The above formulation of the RCC8 relations can be extended to approximate regions. As already discussed in the syntactic generalization of RCC5 relations we perform the following three steps: Assuming definitions of relations between regions exclusively based on the meet operation we, firstly, replace in the definitions of the RCC8 relations (Section 3.2) the variables ranging over regions by variables ranging over approximations and secondly we replace the meet operation, $\wedge$, between regions by the greatest minimal operation, $\underline{\wedge}$ between approximations, and thirdly we replace $\wedge$ by the least maximal operation, $\overline{\wedge}$. Let X and Y be boundary sensitive approximations of regions x and y. The generalized scheme has the form

$$((X\,\underline{\wedge}\,Y \not\approx \bot,\, X\,\underline{\wedge}\,Y \approx X,\, X\,\underline{\wedge}\,Y \approx Y),$$
$$(X\,\overline{\wedge}\,Y \not\approx \bot,\, X\,\overline{\wedge}\,Y \approx X,\, X\,\overline{\wedge}\,Y \approx Y))$$

where

$$X \underline{\wedge} Y \not\approx \perp = \begin{cases} \mathbf{T} & X \underline{\wedge} Y \neq \perp \\ \mathbf{M} & X \underline{\wedge} Y = \perp \text{ and } \delta X \underline{\wedge} \delta Y \neq \perp \\ \mathbf{F} & X \underline{\wedge} Y = \perp \text{ and } \delta X \underline{\wedge} \delta Y = \perp \end{cases} \tag{6}$$

and where

$$X \underline{\wedge} Y \approx X = \begin{cases} \mathbf{T} & X \cong Y \text{ or } (X \ncong Y \text{ and } X \underline{\wedge} Y = X \text{ and} \\ & \delta X \underline{\wedge} \delta Y = \perp) \\ \mathbf{M} & X \ncong Y \text{ and } X \underline{\wedge} Y = X \text{ and } \delta X \underline{\wedge} \delta Y \neq \perp \\ \mathbf{F} & X \underline{\wedge} Y \neq X \end{cases} \tag{7}$$

and where

$$X \underline{\wedge} Y \approx Y = \begin{cases} \mathbf{T} & X \cong Y \text{ or } (X \ncong Y \text{ and } X \underline{\wedge} Y = Y \text{ and} \\ & \delta X \underline{\wedge} \delta Y = \perp) \\ \mathbf{M} & X \ncong Y \text{ and } X \underline{\wedge} Y = Y \text{ and } \delta X \underline{\wedge} \delta Y \neq \perp \\ \mathbf{F} & X \underline{\wedge} Y \neq Y \end{cases} \tag{8}$$

and similarly for $X \overline{\wedge} Y \not\approx \perp$, $X \overline{\wedge} Y \approx X$, and $X \overline{\wedge} Y \approx Y$ using $\overline{\wedge}$ instead of $\underline{\wedge}$ and $\overline{\simeq}$ instead of $\underline{\simeq}$. The formula $X \underline{\simeq} Y$ is true if and only if $X \underline{\wedge} Y = X$ and $X \underline{\wedge} Y = Y$. The formula $X \overline{\simeq} Y$ is true if and only if $X \overline{\wedge} Y = X$ and $X \overline{\wedge} Y = Y$. These definitions correspond to the definition $x = y$ if and only if $x \wedge y = x$ and $x \wedge y = y$ in Section 3.2. In this context the bottom element, $\perp$, is either the value **no** or the function from $G \times G$ to Ω_5 which takes the value **no** for every element of $G \times G$.

Each of the above triples defines a RCC8 relation, so the relation between X and Y can be measured by a pair of RCC8 relations. These relations will be denoted by $\underline{R^8}(X, Y)$ and $\overline{R^8}(X, Y)$. Let X and Y be boundary sensitive approximations:

> THEOREM 4. *The pairs* $(\underline{R^8}(X, Y), \overline{R^8}(X, Y))$ *which can occur are all pairs* (a, b) *where* $a \leq b$ *with the exception of* (**TPP**, **EQ**), (**TPPi**, **EQ**), (**NTPP**, **EQ**), (**NTPPi, EQ**), (**EC**, **TPP**), (**EC, TPPi**), *and* (**EC, EQ**).

Proof (1) We first show that $\underline{R^5}(X, Y) \leq \overline{R^5}(X, Y)$ where $X, Y \in \Omega_5^{G \times G}$, $\underline{R^5}(X, Y)$ and $\overline{R^5}(X, Y)$ are defined using Equation 1. The structure of the argument corresponds to the proof of Theorem 1. We simply use the boundary sensitive operation tables in Section 2.3.2. Consequently, we have $(a \leq b)$ if a and b are refinements of distinct RCC5 relations.

Assume that a and b are refinements of the same RCC5 relation. We need to distinguish the refinement of **DR** and the refinements of **PP** and **PPi**. Refinement of **DR**: $a \leq b$ iff $\delta(X) \underline{\wedge} \delta(Y) \neq \perp \leq \delta(X) \overline{\wedge} \delta(Y) \neq \perp$ (by Definition 6). Consider an arbitrary boundary segment (g_i, g_j). Depending on the values of $\pi(X, (g_i, g_j))$, $\pi^\delta(X, (g_i, g_j))$, $\pi(Y, (g_i, g_j))$, and $\pi^\delta(Y, (g_i, g_j))$, either case 2 or case 3 on page 19 applies. If case 2 applies to (g_i, g_j) then the conditions in 4 are checked. We have $\delta(X) \underline{\wedge} \delta(Y) \neq \perp \leq \delta(X) \overline{\wedge} \delta(Y) \neq \perp$ since we have

$$\pi^\delta(X, (g_i, g_j)) \underline{\wedge} \pi^\delta(Y, (g_i, g_j)) \leq \pi^\delta(X, (g_i, g_j)) \overline{\wedge} \pi^\delta(Y, (g_i, g_j))$$

by the definitions of $\underline{\wedge}$ and $\overline{\wedge}$. This can be verified using the operation tables in Section 2.2.3. If case 3 applies to (g_i, g_j) then $\pi(X(g_i, g_j)) > \pi^\delta(X(g_i, g_j))$ or $\pi(Y(g_i, g_j)) > \pi^\delta(Y(g_i, g_j))$. We may assume without loss of generality that $\pi(X(g_i, g_j)) > \pi^\delta(X(g_i, g_j))$ and $\pi(X(g_i, g_j)) \neq \perp$. In this case we have $\pi^\iota(X, (g_i, g_j)) \neq \perp$ and $X(g_i, g_j) \geq$ **pbo** and $X(g_j, g_i) \geq$ **pbo** by the definitions of π, π^δ, and π^ι. If case 3 applies to (g_i, g_j) then the conditions in 5 are checked. Now, let us assume that $\pi^\delta(X, (g_i, g_j)) \overline{\wedge}\ \pi^\delta(Y, (g_i, g_j)) \neq \perp$. Consequently we have $Y(g_i, g_j) \geq$ **pbo** or $Y(g_j, g_i) \geq$ **pbo**. Since we have **pbo** $\overline{\wedge}$ **pbo** $\neq \perp$ we must have that $(X(g_i, g_j)) \overline{\wedge} (Y(g_i, g_j)) \neq \perp$ or that $(X(g_j, g_i)) \overline{\wedge} (Y(g_j, g_i)) \neq \perp$, and $X \overline{\wedge} Y \neq \perp$. This contradicts $X \overline{\wedge} Y = \perp$ which holds due to the assumption that a and b are refinements of **DR**. Consequently we have $\pi^\delta(X, (g_i, g_j)) \overline{\wedge} \pi^\delta(Y, (g_i, g_j)) = \perp$. We had $\delta(X) \underline{\wedge} \delta(Y) \neq \perp =$ **F** if all (g_i, g_j) were of type 3 and we have $\delta(X) \underline{\wedge} \delta(Y) \neq \perp \leq \delta(X) \overline{\wedge} \delta(Y) \neq \perp$ in the general case.

Refinement of **PP**: $a \leq b$ iff $\delta(X) \underline{\wedge} \delta(Y) \neq \perp \geq \delta(X) \overline{\wedge} \delta(Y) \neq \perp$ (by Definitions 7 and 8). Following the line of argument above one can see this for (g_i, g_j) where case 3 applies and the conditions in 5 are checked since $\pi^\delta(X, (g_i, g_j)) \overline{\wedge} \pi^\delta(Y, (g_i, g_j)) \geq \pi^\delta(X, (g_i, g_j)) \underline{\wedge} \pi^\delta(Y, (g_i, g_j))$. If case 2 applies to (g_i, g_j) then we have $\pi(X(g_i, g_j)) = \pi^\delta(X(g_i, g_j))$ and $\pi(Y(g_i, g_j)) = \pi^\delta(Y(g_i, g_j))$. Assume that $\pi(X(g_i, g_j)) \neq \perp$ and that $\pi(Y(g_i, g_j)) \neq \perp$ which is the only case where the outcome of $\underline{\wedge}$ could differ from the outcome of $\overline{\wedge}$. Since we also have $X \underline{\wedge} Y = X$ (by refinement of **PP**) we have *if* $(X(g_i, g_j)) \neq \perp$ *then* $(Y(g_i, g_j)) \geq$ **fbo**. Consequently, we have $(\pi^\delta(X, (g_i, g_j)) \underline{\wedge} \pi^\delta(Y, (g_i, g_j))) = (\pi^\delta(X, (g_i, g_j)) \overline{\wedge} \pi^\delta(Y, (g_i, g_j)))$ for all (g_i, g_j) of type 2. Thus $\delta(X) \underline{\wedge} \delta(Y) \neq \perp \geq \delta(X) \overline{\wedge} \delta(Y) \neq \perp$. The argument for the refinement of **PPi** is similar and omitted here.

(2) The cases (**TPP**, **EQ**), (**TPPi**, **EQ**), (**NTPP**, **EQ**), (**NTPPi**, **EQ**) cannot occur since these are refinements of (**PP**, **EQ**) and (**PPi**, **EQ**), which cannot occur by Theorem 1.

(3) Consider the cases (**EC**, **TPP**) and (**EC**, **EQ**). If $\overline{R^8}$ = **TPP** or $\overline{R^8}$ = **EQ** then for arbitrary (g_i, g_j) we have if $(X(g_i, g_j)) \neq \perp$ then $(X(g_i, g_j)) \overline{\wedge}\ (Y(g_i, g_j)) \neq \perp$ and if $\pi^\delta(X, (g_i, g_j)) \neq \perp$ then we have that $\pi^\delta(X, (g_i, g_j)) \overline{\wedge} \pi^\delta(Y, (g_i, g_j)) \neq \perp$. Thus, if $(X(g_i, g_j)) \geq$ **pbo** then $(Y(g_i, g_j)) \geq$ **pbo**. Since we assume $\underline{R^8}$ = **EC** we have $(X(g_i, g_j)) \underline{\wedge} (Y(g_i, g_j)) = \perp$, i.e., $\max((X(g_i, g_j)), (Y(g_i, g_j)))$ = **pbo** by definition of

Table 3. Possible pairs of minimal and maximal relations (The relations **TPPi** and **NTPPi** are omitted)

$\underline{R}^8 \backslash \overline{R^8}$	DC	EC	PO	TPP	NTPP	EQ
DC	{DC}	{DC, EC}	{DC, EC, PO}	{DC, EC, PO, TPP}	{DC, EC, PO, TPP, NTPP}	{DC, EC, PO, TPP, NTPP, EQ}
EC	(1)	{EC}	{EC, PO}	(3)	(3)	(3)
PO	(1)	(1)	{PO}	{PO, TPP}	{PO, TPP, NTPP}	{PO, TPP, NTPP, EQ}
TPP	(1)	(1)	(1)	{TPP}	{TPP, NTPP}	(2)
NTPP	(1)	(1)	(1)	(1)	{NTPP}	(2)
EQ	(1)	(1)	(1)	(1)	(1)	{EQ}

$\underline{\triangle}$. Without loss of generality consider $(X(g_i, g_j))$ = **pbo** and $(Y(g_i, g_j))$ = **pbo**. We have $\pi^\delta(X, (g_i, g_j))$ = **po** and $\pi^\delta(Y, (g_i, g_j))$ = **po** and $\pi^\delta(X, (g_i, g_j)) \underline{\triangle}\ \pi^\delta(Y, (g_i, g_j)) = \perp$ and $\underline{R^8} \neq$ **EC** which contradicts the assumption. Consequently the cases (**EC**, **TPP**) and (**EC**, **EQ**) cannot occur. The argument for (**EC**, **TPPi**) is similar and omitted here.

A Haskell [28] program generating all remaining cases can be obtained from the authors.

5.3. *Correspondence of syntactic and semantic generalization*

Let X and Y be boundary sensitive approximations with respect to the partition G.

LEMMA 5. *If there are* $g_i, g_j \in G$ *such that* $(X(g_i, g_j)) = \omega_i$ *and* $(Y(g_i, g_j)) = \omega_2$ *with* $\max(\omega_i, \omega_2)$ = **fbo** *and* $\min(\omega_i, \omega_2) \geq$ **pbo**, *then* $\min(SEM(X, Y))$ = **PO**.

Proof Assume ω_i = **pbo** and ω_2 = **fbo**. By definition of $(X\ (g_i, g_j))$ = **pbo** and $(Y\ (g_i, g_j))$ = **fbo** all $x \in \|X\|$ and $y \in \|Y\|$ overlap at g_i, i.e., $(x \wedge y) \wedge g_i \neq \perp$, and, hence, $X \underline{\triangle} Y \neq \perp$ and $\underline{R}^8(X, Y) \geq$ **PO**. The values of ω_i and ω_2 are also consistent with the existence of $x \in \|X\|$ and $y \in \|Y\|$ such that $x \wedge g_i \neq g_i$, $y \wedge g_i \neq g_i$, and $(x \vee y) \wedge g_i = g_i$. Consequently, there are $x \in \|X\|$ and $y \in \|Y\|$ such that $x \wedge y \neq x$ and $x \wedge y \neq y$ and hence **PO**$(x, y) \in SEM(X, Y)$ and **PO** = $\min(SEM(X, Y))$. The same argument holds for ω_i = **fbo** and ω_2 = **pbo** and for ω_i = **fbo** and ω_2 = **fbo**.

We define the semantically corrected syntactic generalization of RCC8 as:

$$SYN(X, Y) = (\underline{R_C^8}(X, Y), \overline{R^8}(X, Y))$$

where $\underline{R_C^8}(X, Y) = \underline{R^8}(X, Y)$ if there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = \omega_i$ and $\overline{(Y(g_i, g_j))} = \omega_2$ with $\max(\omega_i, \omega_2) =$ **fbo** and $\min(\omega_i, \omega_2) \geq$ **pbo** and $\underline{R_C^8}(X, Y) = \underline{R^8}$ otherwise.

The semantic generalization of RCC8 relations is defined as

$$SEM(X, Y) = \{\rho \in \text{ RCC8 } \mid \underline{R_C^8}(X, Y) \leq \rho \leq \overline{R^8}(X, Y)\},$$

where RCC8 is the set of boundary sensitive binary topological relations and $\leq$ is the ordering of the RCC8 lattice.

THEOREM 6. *For any boundary sensitive approximations X and Y of regular spatial regions, the two ways of measuring the relationship of X to Y are equivalent in the sense that SYN(X, Y) = SEM(X, Y)*

In order to prove this theorem we need the following two lemmata:

LEMMA 7. $\delta(X) \underline{\wedge} \delta(Y) \neq \perp$ *if and only if there are* $x \in \|X\|$ *and* $y \in \|Y\|$ *such that (if* $(x, y) \in \min\{\rho(x, y) \mid \rho \in RCC8\}$[8] *then there are partition cells* g_i *and* g_j *such that* $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \perp)$.

Proof ($\Rightarrow$) If $\delta(X) \underline{\wedge} \delta(Y) \neq \perp$ then there are cells g_i and g_j such that: (a) $\pi(X(g_i, g_j)) = \pi^\delta(X(g_i, g_j))$ and $\pi(Y(g_i, g_j)) = \pi^\delta(Y(g_i, g_j))$ and $\pi^\delta(X(g_i, g_j)) \underline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \perp$ (Definitions 2 and 4); or (b) $(\pi(X(g_i, g_j)) > \pi^\delta(X(g_i, g_j))$ or $\pi(Y(g_i, g_j)) > \pi^\delta(Y(g_i, g_j)))$ and $\pi^\delta(X(g_i, g_j)) \overline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \perp$ (Definitions 3 and 5). If case (a) holds then we have $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \perp)$ for all $x \in \|X\|$ and $y \in \|Y\|$ by definition of $\underline{\wedge}$. If case (b) holds then there some $x \in \|X\|$ and $y \in \|Y\|$ such that $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \perp)$ by definition of $\overline{\wedge}$. The outcome of $\delta(X) \underline{\wedge} \delta(Y) \neq \perp$ provides a refinement of the relation $\underline{R^5}$ that is minimal with respect to the RCC5 classification (Theorem 2). We need to consider the refinement of the RCC5 relations, i.e., five cases. There is no refinement for the relations **PO** and **EQ**. In the case **EQ** we have boundary intersection since $\delta(x) = \delta(y)$. In the case of **PO** we have $\delta(x) \wedge \delta(y) \neq \perp$ or $\delta(x) \cap \delta(y) \neq \perp$[9] since we have if **PO**(x, y) then $x \wedge y \neq \perp$ and $x \wedge y \neq x$ and $x \wedge y \neq y$. The refinement of **PPi** is similar to the refinement of **PP** and will not be considered separately. It remains to discuss the refinement of **DR** and **PP**: In the case of $\underline{R^5} =$ **DR** we have $X \underline{\wedge} Y = \perp$ and due to the constraints (b) either $\pi(X(g_i, g_j)) =$ **fo** and $\pi^\iota(X(g_i, g_j)) =$ **po** and $\pi(Y(g_i, g_j)) = \pi^\delta(Y(g_i, g_j)) =$ **po**, or $\pi(Y(g_i, g_j)) =$ **fo** and $\pi^\iota(Y(g_i, g_j)) =$ **po** and $\pi(X(g_i, g_j)) = \pi^\delta(X(g_i, g_j)) =$ **po** holds. Consequently, all regions

$x \in \|X\|$, $y \in \|Y\|$ with $(x, y) \in$ **DR** have boundary intersection, i.e., rm $\min\{\rho(x, y) \mid \rho \in RCC8\} =$ **EC**. Consequently, the minimal relation between x and y involves boundary intersection. Consider the case of $\underline{R}^5 =$ **PP**. Due to constraint (b) there are regions $x \in \|X\|$, $y \in \|Y\|$ with $\delta x \wedge \delta y \neq \bot$ and, hence, **TPP**(x, y). Since **PP** refines to **TPP** and **NTPP** and **TPP** $<$ **NTPP** the minimal relation between x and y involves boundary intersection.

($\Leftarrow$) For approximations X and Y and boundary segments (g_i, g_j) one of the following cases can occur:

i $\pi^\delta(X(g_i, g_j)) \underline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \bot$;

ii $\pi(X(g_i, g_j)) = \pi^\delta(X(g_i, g_j))$ and $\pi(Y(g_i, g_j)) = \pi^\delta(Y(g_i, g_j))$ and $\pi^\delta(X(g_i, g_j)) \underline{\wedge} \pi^\delta(Y(g_i, g_j)) = \bot$ and $\pi^\delta(X(g_i, g_j)) \overline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \bot$;

iii $\pi(X(g_i, g_j)) > \pi^\delta(X(g_i, g_j))$ or $\pi(Y(g_i, g_j)) > \pi^\delta(Y(g_i, g_j))$ and $\pi^\delta(X(g_i, g_j)) \underline{\wedge} \pi^\delta(Y(g_i, g_j)) = \bot$ and $\pi^\delta(X(g_i, g_j)) \overline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \bot$;

iv $\pi^\delta(X(g_i, g_j)) \overline{\wedge} \pi^\delta(Y(g_i, g_j)) = \bot$.

In case (i) we have for all $x \in \|X\|$ and for all $y \in \|Y\|$: $\delta(x) \wedge \delta(y) \wedge (g_i \wedge g_j) \neq \bot$. In case (ii) and (iii) we can always find a region $x \in \|X\|$ and a region $y \in \|Y\|$ such that $\delta(x) \wedge \delta(y) \wedge (g_i \wedge g_j) \neq \bot$. In case (iv) we have for all $x \in \|X\|$ and for all $y \in \|Y\|$: $\delta(x) \wedge \delta(y) \wedge (g_i \wedge g_j) = \bot$. Let $\beta_{X,Y} : G \times G \to \{\text{i, ii, iii, iv}\}$ be a function that returns for each single (g_i, g_j) a symbol indicating which of the above cases applies given the approximations X and Y and let $L_{X,Y} = \{\beta_{X,Y}(g_i, g_j) \mid (g_i, g_j) \in G \times G, g_i \neq g_j\}$ be the set of the cases that do actually occur with respect to all boundary segments (g_i, g_j). Assume that there are regions $x \in \|X\|$ and $y \in \|Y\|$ such that $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \bot)$. In this case i $\in L_{X,Y}$, ii $\in L_{X,Y}$, or iii $\in L_{X,Y}$ holds. If i $\in L_{X,Y}$ holds then we have $\delta X \underline{\wedge} \delta Y \neq \bot$. If i $\notin L_{X,Y}$ and iii $\in L_{X,Y}$ holds then case 3 (page 19) applies and we have $\delta X \underline{\wedge} \delta Y \neq \bot$ by Definition 5. If i $\notin L_{X,Y}$ and iii $\notin L_{X,Y}$ and ii $\in L_{X,Y}$ holds then case 2 (page 19) applies and we have $\delta X \underline{\wedge} \delta Y = \bot$ by Definition 4. This corresponds correctly to $\underline{R}^8 =$ **DC**.

LEMMA 8. $\delta(X) \overline{\wedge} \delta(Y) \neq \bot$ *if and only if there are* $x \in \|X\|$ *and* $y \in \|Y\|$ *such that (if* $(x, y) \in \max\{\rho(x, y) \mid \rho \in RCC8\}$ *then there are partition cells* g_i *and* g_j *such that* $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \bot)$.

Proof ($\Rightarrow$) If $\delta(X) \overline{\wedge} \delta(Y) \neq \bot$ then there are cells g_i and g_j such that: (a) $\pi(X(g_i, g_j)) = \pi^\delta(X(g_i, g_j))$ and $\pi(Y(g_i, g_j)) = \pi^\delta(Y(g_i, g_j))$ and $\pi^\delta(X(g_i, g_j)) \overline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \bot$ (Definitions 2 and 4); or (b) $(\pi(X(g_i, g_j)) > \pi^\delta(X(g_i, g_j))$ or $\pi(Y(g_i, g_j)) > \pi^\delta(Y(g_i, g_j)))$ and $\pi^\delta(X(g_i, g_j)) \underline{\wedge} \pi^\delta(Y(g_i, g_j)) \neq \bot$ (Definitions 3 and 5). If case (b) holds then we have $(\delta g_i \wedge \delta g_j) \wedge \delta x \wedge \delta y \neq \bot)$ for all $x \in \|X\|$ and $y \in \|Y\|$

by definition of $\underline{\wedge}$. If case (a) holds then there are $x \in \|X\|$ and $y \in \|Y\|$ such that $\delta x \wedge \delta y \wedge (\delta g_i \wedge \delta g_j) \neq \perp$. Corresponding to Lemma 7 we need to consider the refinement of the RCC5 relations $\overline{R^5}$. The treatment of **PO**, **EQ**, and **PPi** is similar to Lemma 7 and omitted here. It remains to discuss $\overline{R^5} = \mathbf{DR}$ and $\overline{R^5} = \mathbf{PP}$. Consider $\overline{R^5} = \mathbf{DR}$: Since $\delta(X) \overline{\wedge} \delta(Y) \neq \perp$ the relation **DR** refines to **EC** (by the $\overline{\wedge}$ version of Definition 6) and since $\mathbf{EC} > \mathbf{DC}$ the greatest relation that can hold between x and y involves boundary intersection at a boundary segment (g_i, g_j). If $\overline{R^5} = \mathbf{PP}$ then we need to distinguish two cases: $X(g_i, g_j) \underline{\wedge} Y(g_i, g_j) \neq X(g_i, g_j)$ and $X(g_i, g_j) \underline{\wedge} Y(g_i, g_j) = X(g_i, g_j)$. If $X(g_i, g_j) \underline{\wedge} Y(g_i, g_j) \neq X(g_i, g_j)$ then the regions $x \in \|X\|$ and $y \in \|Y\|$ with $\mathbf{PP}(x, y)$ are those with $\delta x \wedge \delta y \wedge (\delta g_i \wedge \delta g_j) \neq \perp$ (because of (a)). Consequently, the greatest relation that can hold between x and y involves boundary intersection at (g_i, g_j). If $X(g_i, g_j) \underline{\wedge} Y(g_i, g_j) = X(g_i, g_j)$ then all $x \in \|X\|$ and $y \in \|Y\|$ have boundary intersection at (g_i, g_j).

($\Leftarrow$) Let $L_{X,Y}$ be defined as in Lemma 7. If i $\in L_{X,Y}$ holds then we have $\delta X \overline{\wedge} \delta Y \neq \perp$. If i $\notin L_{X,Y}$ and ii $\in L_{X,Y}$ holds then case 2 (page 19) applies and we have $\delta X \overline{\wedge} \delta Y \neq \perp$ by Definition 4. If i $\notin L_{X,Y}$ and ii $\notin L_{X,Y}$ and iii $\in L_{X,Y}$ holds then case 3 (page 19) applies and we have $\delta X \overline{\wedge} \delta Y = \perp$ by Definition 5. This corresponds correctly to $\overline{R^8} = \mathbf{NTPP}$.

Now we have all the material required for the proof of Theorem 6.

Proof Corresponding to the proof of Theorem 2 there are three things to demonstrate. Firstly that for all $x \in \|X\|$, and $y \in \|Y\|$, that $\underline{R^8_C}(X, Y) \leq \rho(x, y)$. Secondly, for all x and y as before, that $\rho(x, y) \leq \overline{R^8}(X, Y)$, and thirdly that if ρ is any RCC8 relation such that $\underline{R^8_C}(X, Y) \leq \rho \leq \overline{R^8}(X, Y)$ then there exist particular x and y which stand in the relation ρ to each other.

Firstly. We need to consider two cases: (i) There are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = \omega_1$ and $(Y(g_i, g_j)) = \omega_2$ with $\max(\omega_1, \omega_2) = \mathbf{fbo}$ and $\min(\omega_1, \omega_2) \geq \mathbf{pbo}$. In this case we have $\underline{R^8_C}(X, Y) = \mathbf{po} \leq \rho(x, y)$ by Lemma 5. (ii) Otherwise: In this case it is necessary to consider each of the three components $X \underline{\wedge} Y \not\approx \perp$, $X \underline{\wedge} Y \approx X$, and $X \underline{\wedge} Y \approx Y$. If $X \underline{\wedge} Y \not\approx \perp > \mathbf{F}$ then we have to show that (a) for all $x \in \|X\|$ and $y \in \|Y\|$ that $x \wedge y \neq \perp > \mathbf{F}$ and (b) if $X \underline{\wedge} Y \not\approx \perp > \mathbf{M}$ then for all $x \in \|X\|$ and $y \in \|Y\|$ that $x \wedge y \neq \perp > \mathbf{M}$. (a) is a consequence of Theorem 2. (b) is a consequence of Theorem 2 and Lemma 7. If $X \underline{\wedge} Y \approx X > \mathbf{F}$ then we have to show that (a) for all $x \in \|X\|$ and $y \in \|Y\|$ that $x \wedge y \approx x > \mathbf{F}$ and (b) if $X \underline{\wedge} Y \approx X > \mathbf{M}$ then for all $x \in \|X\|$ and $y \in \|Y\|$ that $x \wedge y \approx x > \mathbf{M}$. (a) is a consequence of Theorem 2. (b) is a consequence of Theorem 2 and Lemma 7. Similarly for $X \underline{\wedge} Y \approx Y > \mathbf{F}$ and $X \underline{\wedge} Y \approx Y > \mathbf{M}$.

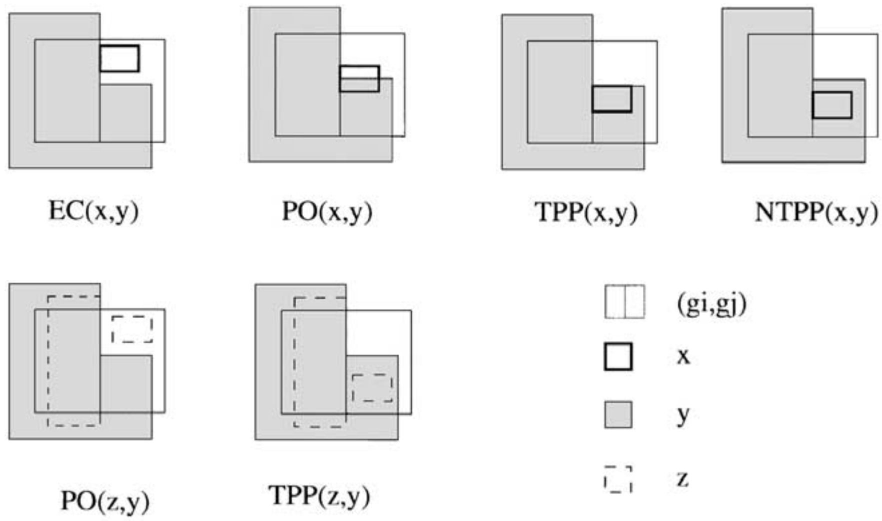


Figure 6. Possible geometric interpretations for (**EC**, **NTPP**) and (**PP**, **TPP**).

Secondly. The proof for $\rho(x, y) \leq \overline{R^8}(X, Y)$ similar and omitted here. It relies on Theorem 2 and Lemma 8.

Thirdly. We have to show that for each RCC8 relation, ρ, where $\underline{R_C^8}(X, Y) \leq \rho \leq \overline{R^8}(X, Y)$, there are $x \in \|X\|$ and $y \in \|Y\|$ such that the relation of x to y is ρ. This is done by considering the various possibilities for $\underline{R_C^8}(X, Y)$ and $\overline{R^8}(X, Y)$. We will only consider the cases (**EC**, **NTPP**) and (**PO**, **TPP**) but the others are similar. If $\underline{R_C^8}(X, Y) = \textbf{EC}$ and $\overline{R^8}(X, Y) = \textbf{NTPP}$ then the following constraints need to be satisfied: $X \underline{\wedge} Y = \perp$, $X \overline{\wedge} Y = X$, $X \overline{\wedge} Y \neq Y$, $\delta X \underline{\wedge} \delta Y \neq \perp$, $\delta X \overline{\wedge} \delta Y = \perp$. The case $\delta X \underline{\wedge} \delta Y > \delta X \overline{\wedge} \delta Y$ can only occur if case (3) holds and definition 5 applies. Due to these constraints we have for all boundary segments (g_i, g_j) if $(\pi^\delta(X, (g_i, g_j)) \neq \perp$ and $\pi^\delta(Y, (g_i, g_j)) \neq \perp)$ then $(\pi(X, (g_i, g_j)) = \textbf{po}$ and $\pi(Y, (g_i, g_j)) > \pi^\delta(Y, (g_i, g_j)))$. This gives us enough freedom to construct regions $x \in \|X\|$ and $y \in \|Y\|$ as sums of parts of partition cells that satisfy these constraints such that the relations **EC**(x, y), **PO**(x, y), **TPP**(x, y), and **NTPP**(x, y) hold. Examples are given in Figure 6 (configurations **EC**(x, y), **PO**(x, y), **TPP**(x, y), and **NTPP**(x, y)).

Consider the case (**Po**, **TPP**) it can only occur if the $x \in \|X\|$ are complex regions. Examples are given in Figure 6 (configurations **PO**(z, y) and **TPP**(z, y)).

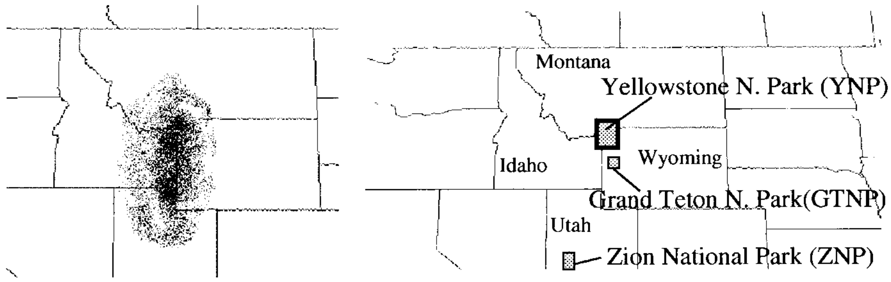


Figure 7. Approximate location of an "area of bad weather" (left) and National Parks (right) within the regional partition formed by the Federal States of the U.S.

6. Discussion

The present paper continues work started in (Bittner and Stell, to appear) where we discussed the relationships between the vagueness of human concepts and the indeterminate character of spatial location of objects to which those concepts apply. We argued that approximate descriptions of the (indeterminate) location of objects subject to vagueness provide means to describe those phenomena in a determinate and formally well defined manner. In those approximate descriptions people refer to location in terms of relationships to a regional partition of the underlying space. Formally those approximations are based our model proposed in (Bittner and Stell 1998).

Consider, for example, the regional partition formed by the 50 constituent states of the United States of America. A fragment of this partition is presented in the left and right parts of Figure 7. In the foreground of the left part of the figure we see in addition an area of bad weather, represented by a dark dotted region. In the foreground of the right part we see National Parks.

The area of bad weather is subject to vagueness in the sense that the location of its boundaries is indeterminate. Wherever the boundaries might be located, they certainly lie skew to the boundaries of the relevant states. But the figure also indicates that there are parts of the area of bad weather that are also parts of Wyoming, others which are parts of Montana, others which are parts of Utah, and yet others which are parts of Idaho. These relationships are not affected by the indeterminacy of the boundary location (Bittner and Smith 2001b) argue that when making judgments about the location of the "area of bad weather" we, the judging subjects, then deliberately employ this partition as our frame of reference in order to describe approximate location in a determinate manner.

Our approach of using approximations for dealing with indeterminacy of location that was caused by vagueness of the underlying concepts differs from the standard approach to vagueness which are based on supervaluation (van Fraassen 1966; Fine 1975). Examples are (Varzi 2001; Smith and Brogaard 2001). Relationships between the supervaluation and the approximation based approaches are discussed in (Bittner and Smith 2001b).

The regional partitions underlying the approximations discussed in this paper are special forms of granular partitions (Smith and Brogaard, to appear; Bittner and Smith 2001a). Granular partitions are understood as ways of dividing up or structuring reality in order to make it more easily graspable by cognitive subjects such as ourselves. Partitions underlying approximations are special in the sense that they serve as frames of reference. The types of granular partitions that are used as frames of reference characteristically have the following properties: (1) they are relatively stable, i.e., they do not change over time (we can also demand that they are specifiable in some easily communicable way); and (2) the regions completely carve up space in the sense that the there is no “no-mans-land”, i.e., as a set they are jointly exhaustive and pair-wise disjoint. The second point has the consequence that the regions forming the partition share boundary segments. This property is critical for making boundary-sensitive approximations.

In the present paper we go beyond the approximation of single objects. We consider which mereological (RCC5) and which mereo-topological (RCC8) relations can hold between objects which are approximated with respect to the same underlying regional partition. Consider Figure 7. Assume that we have descriptions of the approximate location of the “area of bad weather” (left) and of the approximate locations of the National Parks Yellowstone, Grand Teton, and Zion (right). Using the formalism presented in this paper we then are able to determine which relations hold among all those objects. In particular we are able to determine whether the “area of bad weather” overlaps Yellowstone park or not.

In present paper we distinguish between boundary insensitive and boundary sensitive approximations. Boundary sensitive approximations are of particular importance since they reflect the fact that bona fide boundaries correspond to discontinuities (and thus salient features) in the underlying reality (Smith 1995). They take the relationships of the approximated objects with those boundaries explicitly into account.

Cognitive evidence that indicates that people actually do perform reasoning about approximations within regional partitions can be found for example in (Stevens and Coupe 1978). In this context we assume that those findings about reasoning about cardinal directions generalize to reasoning about topological relations.

Justification for the focus on boundary-sensitive approximations whenever the underlying regional partition is aligned to bona fide boundaries comes from three directions: (i) As shown, for example, in (Smith 1995) and (Smith 1997) boundaries are important features of the ontological makeup of reality. From this perspective it seems to be reasonable to assume that those ontologically salient features are reflected in (approximate) descriptions of reality; (ii) Focus on boundaries is also supported by the fact that research on understanding human vision indicates the important nature of boundaries for the human visual system; (iii) Experiments published in (Knauff et al. 1997) have shown that humans prefer (if given the choice) to refer to boundary sensitive relations, i.e., the RCC9-relations, rather than to refer to boundary insensitive relations, i.e., the RCC5 relations.

7. Conclusions and further work

Approximate qualitative spatial reasoning is based on:

1. Jointly exhaustive and pair-wise disjoint sets of qualitative relations between exact regions, which are defined in terms of the meet operation of the underlying Boolean algebra structure of the domain of regions. As a set these relations must form a lattice with bottom and top element.
2. Approximations of regions with respect to a regional partition of the underlying space. Semantically, an approximation corresponds to the set of regions it approximates.
3. Pairs of meet operations on those approximations, which approximate the meet operation on exact regions.

Based on those "ingredients" syntactic and semantic generalizations of jointly exhaustive and pair-wise disjoint relations between exact regions were defined. Generalized relations hold between approximations of regions rather than between (exact) regions themselves. Syntactic generalization is based on replacing the meet operation defining relations between exact regions by its minimal and maximal counterparts on approximations. Semantically, syntactic generalizations yield upper and lower bounds (within the underlying lattice structure) on relations that can hold between the corresponding approximated exact regions.

There is considerable scope for further work building on the results in this paper. We have assumed that the regions being approximated are precisely known regions in a continuous space. However, there are practical examples where approximate regions are themselves approximated. This can occur when spatial data is required at several levels of detail, and the less detailed representations are approximations of the more detailed ones. Thus one direction for future investigation is to extend the techniques in this paper to the case where the regions being approximated are discrete, rather than continuous. This could make use of the algebraic approach to qualitative discrete space presented in (Stell 2000). Another direction of ongoing research is to apply techniques presented in this paper to the temporal domain (Bittner, to appear).

Acknowledgements

The first author was supported by DARPA under the Rapid Knowledge Formation program and the National Science Foundation under the Research on Learning and Education program. The second author was supported by EPSRC under grant GR/M 56685 "Managing Vagueness, Uncertainty and Granularity in Spatial Information Systems". This support is gratefully acknowledged.

Notes

[1] Notice that $x = y$ is implied by ($x \wedge y = x$ and $x \wedge y = y$) and, hence, can be expressed in terms of $\wedge$.

[2] Notice that, given that x and y are 2-dimensional regions, then their meet is empty, $x \wedge y = \bot$, if only their boundaries intersect since the result of this intersection is not a 2-dimensional region. The intersection of 1-dimensional regions is empty even if they intersect in a single or multiple disconnected points.

[3] Consider Figure 1. An intersection of the boundaries of the regions z and u occurs at the boundary segment shared by the cells (1, 1) and (1, 2), i.e., $(\delta(1,1) \wedge \delta(1,2)) \wedge (\delta z \wedge \delta u) \neq \bot$.

[4] We use the notion of an ordered pair, (g_i, g_j), to refer to the boundary segment shared by the partition cells g_i and g_j. This slightly conflicts with the usage of (g_i, g_j) as argument of the approximation function α_5, e.g., in $(X\,(g_i, g_j))$, where it refers to the cells themselves. The context should make clear which interpretation is intended.

[5] Remember Section 2.3.

[6] Notice that regions do not occur in the formula below since here we are at the approximation or syntactic level where the notation $x \in \|X\|$ refers to the intended semantic interpretation.

[7] In fact this is the maximal minus operation $\ominus_{max}$ the corresponding minimal operation $\ominus_{min}$ would yield $\mathbf{po} \ominus_{min} \mathbf{po} = \mathbf{po}$.

[8] Assuming the ordering of the RCC8 lattice.

[9] $\delta(x) \wedge \delta(y) = \bot$ if there is only a zero-dimensional intersection of $\delta(x)$ and $\delta(y)$.

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