Topological Relationships of Complex Points and Complex Regions

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Abstract. Topological relationships between spatial objects have been a focus of research on spatial data handling and reasoning for a long time. Especially as predicates they support the design of suitable query languages for spatial data retrieval and analysis in databases. Unfortunately, they are so far only applicable to simplified abstractions of spatial objects like single points, continuous lines, and simple regions, as they occur in systems like current geographical information systems and spatial database systems. Since these abstractions are usually not sufficient to cope with the complexity of geographic reality, their generalization is needed which especially has influence on the nature, definition, and number of their topological relationships. This paper partially closes this gap and first introduces very general spatial data types for complex points and complex regions. It then defines the corresponding complete sets of mutually exclusive, topological relationships.

Keywords. Topological predicate, spatial data type, 9-intersection model

1 Introduction

For a long time topological relationships have been a focus of research in disciplines like spatial databases, geographical information systems, CAD/CAM systems, image databases, spatial analysis, computer vision, artificial intelligence, cognitive science, psychology, and linguistics. Topological relationships like *overlap*, *inside*, or *meet* describe purely qualitative properties that characterize the relative positions of spatial objects and that are preserved under continuous transformations such as translation, rotation, and scaling. They exclude any consideration of quantitative measures like distance or direction measures and are associated with notions like adjacency, coincidence, connectivity, inclusion, and continuity. In particular, they are needed for spatial reasoning and in spatial query languages where they are, for instance, employed as part of a filter condition in a query.

Some well known, formal models for the definition of topological relationships have already been proposed (see Section 2.2). But they are essentially tailored to the treatment of simple regions and lines. Simple regions are two-dimensional point sets topologically equivalent to a closed disc, and simple lines are one-dimensional features embedded in the plane with two end points. Points are not taken into account, since their interrelations are trivial. Unfortunately, the variety and complexity of geographic entities can be hardly modeled with these simple geometric structures. Due to a lack of space, we will confine ourselves to points and regions in this paper. With regard to points, we will allow finite collections of single points as point objects. With regard to regions, the two main extensions relate to separations of the exterior (holes) and to separations of the interior (multiple components). Both extensions ensure closure of geometric operations and are common in geographical applications. Countries, e.g., can be made up of multiple components (islands) and can have holes (enclaves).

The goals of this paper are twofold: first we introduce and formalize spatial data types for *complex points* and *complex regions*. Then all possible topological relationships between two complex points and between two complex regions, respectively, are derived from the well known 9-intersection model. For this purpose, we draw up collections of constraints specifying conditions for valid topological relationships and satisfying the properties of *completeness* and *exclusiveness*. The property of completeness ensures a full covering of all topological situations. The property of exclusiveness ensures that two different relationships cannot hold for the same two spatial objects.

The remainder of the paper is organized as follows: Section 2 discusses related work regarding spatial objects and topological relationships. Section 3 summarizes the spatial data model for which topological relationships will be investigated. Section 4 explains the strategy for deriving topological relationships from the 9-intersection model. In Section 5 all topological relationships between complex points are analyzed. Section 6 does the same for complex regions. Finally, Section 7 draws some conclusions and discusses future work.

2 Related Work

In this section we discuss some related work about spatial objects as the operands of topological relationships (respectively corresponding predicates) (Section 2.1) and about topological relationships themselves (Section 2.2).

2.1 Spatial Objects

In the past, numerous data models and query languages for spatial data have been proposed with the aim of formulating and processing spatial queries in databases (e.g., [8, 9]). *Spatial data types* (see [9] for a survey) like *point*, *line*, or *region* are the central concept of these approaches. They provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. Topological predicates operate on instances of these data types, called *spatial objects*. So far, rather simple object structures (like *simple* points, lines, and regions) have been used as arguments of topological predicates. In this paper, we are interested in topological predicates on *complex* spatial objects for two reasons. First, from an application point of view, simple spatial structures are insufficient abstractions of spatial reality. For example, Italy cannot be modeled by a simple region, since it has the Vatican as a hole and comprises islands in the Mediterranean Sea. Second, from a formal point of view, we have to require closure properties for the spatial data types. This means, e.g., that the geometric intersection, union, and difference of two point, two line, or two region objects, respectively, may not leave the corresponding type definition. Similar considerations lead to a generalization of point objects.

We will give formal definitions of these object structures in Section 3. For the definition of a point data type we use set theory. For the definition of a region data type and its topological predicates we employ the point set paradigm and *point set topology* [7]. Regions are modeled as infinite point sets in the Euclidean plane. Point set topology permits to distinguish different *parts* of the point set of a region. Given such a point set, say A, these parts identify its *boundary* ∂A, its *interior* A°, and its *exterior* A^-, which are pairwise disjoint. The union of A° and ∂A corresponds to the *closure* $\overline{A}$ of A. The effect of applying the *interior* operation to a point set is to eliminate dangling points, dangling lines, and boundary parts. The effect of the *closure* operation is to eliminate cuts and punctures by appropriately supplementing points as well as adding the boundary. Hence, it makes sense only to consider point sets A for which $A = \overline{A^\circ}$ holds. This concept of regularity avoids geometric anomalies in regions and leads to so-called *regular closed* point sets respectively regions without degeneracies [10].

2.2 Topological Relationships

An important approach for characterizing topological relationships rests on the so-called 9-*intersection model* [3–5]. This model allows one to derive a complete collection of mutually exclusive topological relationships for each combination of spatial types. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object A with the corresponding components of another object B. Each intersection is tested with regard to the topologically invariant criteria of emptiness and non-emptiness.

$2^9 = 512$ different configurations are possible from which only a certain subset makes sense depending on the *definition* and combination of spatial objects just considered. For each combination of spatial types this means that each of its predicates can be associated with a unique intersection matrix (Table 1) so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness. Topological relationships that have been investigated so far are restricted in the sense that their argument objects are not allowed to have the most general, possible structure. It is just the objective of this paper to give the most general definitions of spatial objects and to identify the topological relationships between them.

Topological relationships have been first investigated for simple regions [2–5]. For two simple regions eight meaningful configurations have been identified which lead to the well known eight predicates called *disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, and *coveredBy*. The 9-intersection model has been extended with further topological invariants (like the dimension of the

$$\begin{pmatrix} A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ \partial A \cap B^\circ \neq \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^- \cap B^\circ \neq \varnothing & A^- \cap \partial B \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

Table 1. The 9-intersection matrix. Each matrix entry is a 1 (*true*) or 0 (*false*).

intersection components, their types (touching, crossing), the number of components) to discover more details about topological relationships (e.g., [2]).

It is surprising that topological predicates on complex regions have so far not been defined. But the definition of these predicates is particularly important for spatial query languages that aim at integrating complex regions having holes and separations. Two works have so far contributed to a definition of topological relationships for more complex regions. In [1] the so-called TRCR (Topological Relationships for Composite Regions) model only allows sets of disjoint simple regions without holes. But topological relationships between composite regions are defined in an ad hoc manner and are not systematically derived from the underlying model. Moreover, the model is only related to but not directly based on the 9-intersection model. In [6] topological relationships of simple regions with holes are considered. Unfortunately, multi-part regions are not permitted. While the authors take the number of components (area without holes, holes) of two regions into account and consider the large number of topological relationships between all component pairs of both regions, we pursue a global approach that is independent of the number of components. Hence, a further goal of this paper is to provide an integrated treatment of holes and separations for regions and to define topological predicates on complex regions in a systematic way.

Topological predicates between simple points are trivial: either two simple points are *disjoint* or they are *equal*. We have a more general view of point objects and consider a complex point as a finite collection of single points. This leads to the necessity of investigating further topological relationships.

3 Spatial Data Model

In this section we strive for a very general definition of complex spatial objects in the Euclidean plane $\mathbb{R}^2$. The task is to identify those point sets that are admissible for complex point and region objects.

A value of type *point* is defined as a finite set of points in the plane. Thus a type for complex points can be specified as

$$\mathit{point} = \{P \subset \mathbb{R}^2 \mid P \text{ is finite}\}$$

We call a value of this type *complex point.* If $P \in \mathit{point}$ is a singleton set, i.e., $|P| = 1$, P is denoted as a *simple point.* For a simple point p we specify $\partial p = \varnothing$ and $p^\circ = p$, which is the commonly accepted definition. For a complex point $P = p_1, \ldots, p_n$ we then obviously obtain $\partial P = \varnothing$ and $P^\circ = \bigcup_{i=1}^{n} p_i^\circ$.

Since complex regions can be arbitrary points sets but without the geometric anomalies discussed in Section 2.1, we are now already able to give an appropriate definition of a type for *complex regions*:

$$region = \{R \subset \mathbb{R}^2 \mid R \text{ is bounded and regular closed}\}$$

This definition is conceptually somehow "structureless" in the sense that only "flat" point sets are considered and no structural information is revealed. The "structured" view of a regular closed set is that of a region possibly consisting of several area-disjoint components and possibly having area-disjoint holes (Figure 1). Boundary, interior, and exterior result from the corresponding operators on arbitrary point sets.

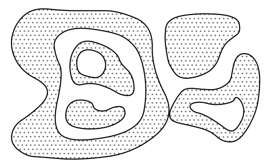


Fig. 1. A complex region.

4 Deriving Topological Relationships from the 9-Intersection Model

Our strategy for the analysis of topological relationships between two complex points or regions is quite simple and yet very general: instead of applying the 9-intersection model to point sets belonging to simple spatial objects, we extend it to point sets belonging to complex spatial objects. Due to the special features of the objects (point, areal properties), the embedding space (here: $\mathbb{R}^2$), the relation between the objects and the embedding space (e.g., it makes a difference whether we consider a point in $\mathbb{R}$ or in $\mathbb{R}^2$), and the employed spatial data model (e.g., discrete, continuous), a number of topological configurations cannot exist and have to be excluded.

Our goal is to determine for each pair of complex spatial data types the corresponding topological constraints or conditions that have to be satisfied; these serve as exclusion criteria for all other impossible configurations. The approach taken starts with the 512 possible matrices and is a two-step process:

(i) For each type combination we give the formalization of a collection of topological constraints for existing relationships in terms of the nine intersections. For each constraint we give reasons for its meaningfulness. The evaluation of each constraint gradually reduces the set of the currently valid matrices by all those matrices not fulfilling the constraint under consideration.
(ii) The existence of topological relationships given by the remaining matrices is verified by realizing prototypical spatial configurations in $\mathbb{R}^2$, i.e., these configurations can be drawn in the plane.

Still open issues relate to the evaluation order, completeness, and minimality of the collection of constraints. Each constraint is a predicate that is matched with all intersection matrices under consideration. All constraints must be satisfied together so that they represent a conjunction of predicates. To say it in other words, constraints are all formulated in conjunctive normal form. Since the conjunction (logical *and*) operator is commutative and associative, the order in which the constraints are evaluated is irrelevant; the final result is always the same. The completeness of the collection of constraints is directly ensured by the second step of the two-step process. The aspect of minimality addresses the possible redundancy of constraints. Redundancy can arise for two reasons. First, several constraints may be correlated in the sense that one of them is more general than the others, i.e., it eliminates at least the matrices excluded by all the other constraints. This can be easily checked by analyzing the constraints themselves and searching for the most non-restrictive and common constraint. Even then the same matrix can be excluded by several constraints simultaneously. Second, a constraint can be covered by some combination of other constraints. This can be checked by a comparison of the matrix collection fulfilling all n constraints with the matrix collection fulfilling $n-1$ constraints. If both collections are equal, then the omitted constraint was implied by the combination of the other constraints and is therefore redundant.

5 Topological Relationships between Complex Points

We now present the constraints for two complex point objects A and B. Each constraint is first formulated colloquially and afterwards formalized by employing the nine intersections. Then a rationale is given explaining why the constraint makes sense. We presuppose that A and B are not empty, because topological relationships for empty operands are not meaningful.

Constraint 1 *All intersections comprising an operand with a boundary operator yield the empty set, i.e.,*

$$\forall\, C \in \{A^\circ, \partial A, A^-\} : C \cap \partial B = \varnothing \ \wedge \ \forall\, D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing$$

Rationale. According to the definition of a complex point $\partial A = \partial B = \varnothing$ holds. The intersection of the empty set with any other component yields the empty set. □

Constraint 2 *The exteriors of two complex point objects always intersect with each other, i.e.,*

$$A^- \cap B^- \neq \varnothing$$

Rationale. We know that $A \cup A^- = \mathbb{R}^2$ and $B \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $A = \mathbb{R}^2$, or (ii) $B = \mathbb{R}^2$, or (iii) $A \cup B = \mathbb{R}^2$. All three situations are impossible, since A, B, and $A \cup B$ are finite sets and $\mathbb{R}^2$ is an infinite set. Thus $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$, and $A \cup B \subset \mathbb{R}^2$ holds. □

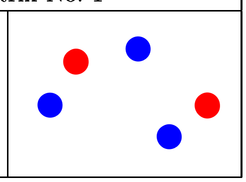
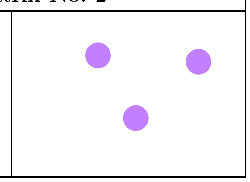
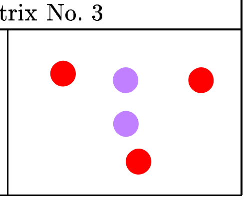
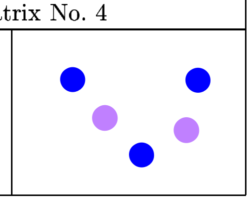
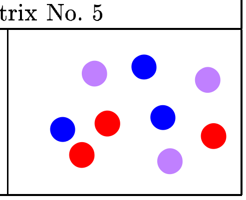
Constraint 3 *Each non-empty part of a complex point intersects at least one non-empty part of the other complex point, i.e.,*

$$(\forall\, C \in \{A^\circ, A^-\} : C \cap B^\circ \neq \varnothing \ \vee \ C \cap B^- \neq \varnothing) \ \wedge$$
$$(\forall\, D \in \{B^\circ, B^-\} : A^\circ \cap D \neq \varnothing \ \vee \ A^- \cap D \neq \varnothing)$$

Rationale. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup B^- = \mathbb{R}^2$. That is, the complex point A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Hence, and because only non-empty object parts are considered, the interior and the exterior of A, respectively B, must intersect at least either the interior or the exterior or both parts of B, respectively A. □

Since $\partial A = \partial B = \varnothing$, the second row and the second column of an intersection matrix only yield empty intersections, and we do not have to consider them any further. The remaining intersections are those in the four corners of a matrix. Hence, in the first and third row and in the first and third column of a matrix at least one "corner" intersection must yield true so that we find the value 1 in the matrix there.

It has been checked with a trivial test program that none of the three constraints can be omitted (otherwise we would obtain more matrices). As a result, we obtain five remaining topological relationships between complex points. The corresponding matrices and their geometric interpretations are given in Table 2.

Matrix No.	Matrix	Illustration
1	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	
4	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
5	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	

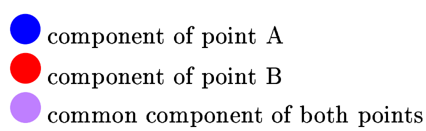


Table 2. The five topological relationship for complex points.

With each matrix we can associate a name for the corresponding topological predicate. Matrix 1 describes the relationship *disjoint*, matrix 2 the relationship *equal*, matrix 3 the relationship *inside*, matrix 4 the relationship *contains*, and matrix 5 the relationship *overlap*.

6 Topological Relationships between Complex Regions

In this section we identify those topological relationships that can be realized between two non-empty, complex regions A and B. We pursue the same strategy as for points and first present constraints that exclude non-existent topological configurations. Note that a *part* of a complex region denotes either its boundary, interior, or exterior and that all parts are non-empty.

Constraint 1 *Each part of a complex region intersects at least one part of the other complex region, i.e.,*

$$(\forall\, C \in \{A^\circ, \partial A, A^-\} : C \cap B^\circ \neq \varnothing \ \vee \ C \cap \partial B \neq \varnothing \ \vee \ C \cap B^- \neq \varnothing) \ \wedge$$
$$(\forall\, D \in \{B^\circ, \partial B, B^-\} : A^\circ \cap D \neq \varnothing \ \vee \ \partial A \cap D \neq \varnothing \ \vee \ A^- \cap D \neq \varnothing)$$

Rationale. We know that $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup \partial B \cup B^- = \mathbb{R}^2$. That is, the complex region A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Hence, each part of A, respectively B, must intersect at least one part of B, respectively A. $\square$

Constraint 2 *Neither the interior nor the exterior of a complex region can be completely contained in or equal to the boundary of the other complex region, i.e.,*

$$\begin{aligned}
& A^\circ \not\subseteq \partial B \ \wedge \ A^- \not\subseteq \partial B \ \wedge \ B^\circ \not\subseteq \partial A \ \wedge \ B^- \not\subseteq \partial A \\
\Leftrightarrow\ & (A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing) \ \wedge \ (A^- \cap B^\circ \neq \varnothing \ \vee \ A^- \cap B^- \neq \varnothing) \ \wedge \\
& (A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^- \cap B^\circ \neq \varnothing) \ \wedge \ (A^\circ \cap B^- \neq \varnothing \ \vee \ A^- \cap B^- \neq \varnothing)
\end{aligned}$$

Rationale. The obvious reason is that the dimension of a boundary with its linear structure is less than the dimensions of the interior and the exterior with their areal structures. The constraint definition shows that the formalization based on subset relationships can be transformed to an equivalent formalization based on the nine intersections. If the interior and the exterior, respectively, of a region is not completely contained in or equal to the boundary of the other region, it intersects either the interior or the exterior or both parts of the other region, and vice versa. $\square$

Constraint 3 *The exteriors of two complex region objects always intersect with each other, i.e.,*

$$A^- \cap B^- \neq \varnothing$$

Rationale. We know that $\overline{A} \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $\overline{A} = \mathbb{R}^2$, or (ii) $\overline{B} = \mathbb{R}^2$, or (iii) $\overline{A} \cup \overline{B} = \mathbb{R}^2$. The situations are all impossible, since A, B, and hence $A \cup B$ are bounded, but $\mathbb{R}^2$ is unbounded. $\square$

Constraint 4 *The boundaries of two complex regions are equal if and only if the interiors and the exteriors, respectively, of both regions are equal, i.e.,*

$$
\begin{aligned}
&(\partial A = \partial B \Leftrightarrow A^\circ = B^\circ \ \wedge \ A^- = B^-) \\
\Leftrightarrow\ &(c \Leftrightarrow d) \Leftrightarrow ((c \ \wedge \ d) \ \vee \ (\neg c \ \wedge \ \neg d)) \text{ where} \\
&c = A^\circ \cap \partial B = \varnothing \ \wedge \ \partial A \cap B^\circ = \varnothing \ \wedge \ \partial A \cap \partial B \neq \varnothing \ \wedge \\
&\quad\ \partial A \cap B^- = \varnothing \ \wedge \ A^- \cap \partial B = \varnothing \text{ and} \\
&d = A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap \partial B = \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \ \wedge \\
&\quad\ \partial A \cap B^\circ = \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing \ \wedge \ \partial A \cap B^- = \varnothing \ \wedge \\
&\quad\ A^- \cap \partial B = \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing
\end{aligned}
$$

Rationale. This very special constraint expresses that complex regions are uniquely characterized by their boundaries. This is ensured by the Jordan Curve Theorem [7]. □

Constraint 5 *If the boundary of a complex region intersects the interior of the other complex region, both its interior and its exterior intersect the interior of the other region, i.e.,*

$$
\begin{aligned}
&((\partial A \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing)) \ \wedge \\
&\ (A^\circ \cap \partial B \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing))) \\
\Leftrightarrow\ &((\partial A \cap B^\circ = \varnothing \ \vee \ (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing)) \ \wedge \\
&\ (A^\circ \cap \partial B = \varnothing \ \vee \ (A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing)))
\end{aligned}
$$

Rationale. On each side of the boundary of a region there is either the region's interior or exterior (Jordan Curve Theorem). On both sides of a line intersecting the interior of this region, we find the interior of the region. If the line is part of the boundary of another region, we obtain the intersection of both regions' interiors and the intersection between the interior of the first region and the exterior of the other region. □

Constraint 6 *If the boundary of a complex region intersects the exterior of the other complex region, both its interior and its exterior intersect the exterior of the other region, i.e.,*

$$
\begin{aligned}
&((\partial A \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \ \wedge \\
&\ (A^- \cap \partial B \neq \varnothing \Rightarrow (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing))) \\
\Leftrightarrow\ &((\partial A \cap B^- = \varnothing \ \vee \ (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \ \wedge \\
&\ (A^- \cap \partial B = \varnothing \ \vee \ (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)))
\end{aligned}
$$

Rationale. The argumentation is similar as for the previous constraint. On each side of the boundary of a region there is either the region's interior or exterior. On both sides of a line intersecting the exterior of this region, we find the exterior of the region. If the line is part of the boundary of another region, we obtain the intersection of both regions' exteriors and the intersection between the interior of the first region and the exterior of the other region. □

Constraint 7 *The boundaries of two complex regions intersect, or the boundary of one region intersects the exterior of the other region, i.e.,*

$$\partial A \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing$$

Rationale. Assuming that the constraint is false. Then neither the boundaries of the two regions nor the boundary of one region and the exterior of the other region intersect. Consequently, according to Constraint 1, each boundary of one region intersects the interior of the other region. Without loss of generality, let us consider a point $p \in A^\circ \cap \partial B$ and an infinite ray s emanating from p in an arbitrary direction. Since the component of A containing p is bounded, s encounters the boundary of A in a point, say, q. This boundary intersects the exterior, the boundary, or the interior of B. According to our assumption the first two cases cannot hold so that q must lie inside the interior of B. We obtain a similar situation as before, except for the fact that now A and B change their roles. We continue to observe the course of s: the ray over and over again alternately encounters a point of $A^\circ \cap \partial B$ and then a point of $\partial A \cap B^\circ$. Since the ray can be prolonged arbitrarily, A and B must be unbounded. But this is a contradiction to the definition of the *region* data type. □

Constraint 8 *If the interiors of two complex regions intersect, the interior of one region also intersects the boundary of the other region, or the regions' boundaries intersect, i.e.,*

$$\begin{aligned} & (A^\circ \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \\ \Leftrightarrow \ & (A^\circ \cap B^\circ = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing) \end{aligned}$$

Rationale. Let us consider a component of the first region and a component of the second region with intersecting interiors. We have to distinguish three situations. First, if the interiors of both components are equal, also their boundaries are equal and hence intersect. Consequently, also the regions' boundaries intersect. Second, if the interiors of both components but not their boundaries intersect, one component is contained in the other. Since this is a proper containment (otherwise the boundaries would intersect), the boundary of one component must be inside the interior of the other component. Consequently, the interior of one region intersects the boundary of the other region. Third, if the interiors and the boundaries of the two components intersect, both conclusions of the constraint hold. □

Constraint 9 *If the interior of a complex region intersects the exterior of the other region, either the interior of the first region intersects the boundary of the second region, or the boundary of the first region intersects the exterior of the second region, or both regions' boundaries intersect, i.e.,*

$$\begin{aligned} & ((A^\circ \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \ \wedge \\ & (A^- \cap B^\circ \neq \varnothing \Rightarrow (\partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing))) \\ \Leftrightarrow \ & ((A^\circ \cap B^- = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing) \ \wedge \\ & (A^- \cap B^\circ = \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \end{aligned}$$

Rationale. If there is an intersection between the interior of a complex region and the exterior of the other complex region, a few different situations for each component causing the intersection can be distinguished. The first situation is that a component partially intersects the interior and the exterior of the other region. Then the boundary of the other region intersects the interior of the first region.

The second situation is that the interior of a component lies completely inside the exterior of the other region. Several cases can now be distinguished. The first case is that also the boundary (and thus the entire component) lies inside and consequently intersects the exterior of the other region. The second case is that the boundary of a component lies only partially inside the exterior of the other region. Again we obtain an intersection between boundary and exterior. The third case is that the boundary of a component intersects the boundary of the other region. Note that the boundary of the component cannot cross the interior of the other region, since then the interior of the component would not be entirely within the exterior of the other region. □

An investigation of the nine constraints with the aid of a trivial test program reveals that all of them are needed with one exception. Constraint 2 is redundant, since the matrices it removes are also eliminated by some combination of the other constraints. Hence it can be omitted. All remaining constraints exclude at least one matrix each. As a result, we obtain 33 topological relationships between complex regions. The corresponding matrices and their geometric interpretation are given in Table 3. The topological relationships between simple regions correspond to the intersection matrices with the numbers 1, 4, 5, 7, 9, 19, 24, and 33.

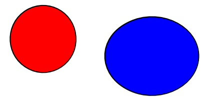
7 Conclusions and Future Work

In this paper we have defined very general spatial data types for complex points and complex regions in the two-dimensional Euclidean plane on the basis of point set theory and point set topology. The increasing complexity of spatial data types leads to a larger variety of topological relationships. The investigation and formalization of complete collections of mutually exclusive topological relationships between complex points and between complex regions, respectively, has been the main contribution of this paper. It has been done on the basis of the well-known 9-intersection model. We have identified 5 binary relationships between complex points and 33 binary relationships between complex regions.

For future work one could analyze possible topological relationships between complex lines. This first necessitates a formal definition of a corresponding type. Similarly, one could investigate the possible relationships between the three pairs of mixed types. Another problem of interest is the large number of relationships. Whereas the 5 relationships between complex points are manageable by the user, this is not the case for the 33 relationships between complex regions. Here one could think about clustering techniques in the sense of [2].

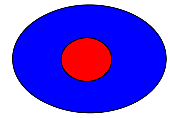
Matrix No. 1

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



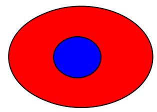
Matrix No. 2

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



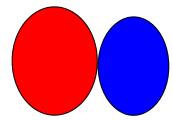
Matrix No. 3

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



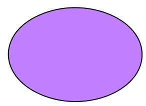
Matrix No. 4

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



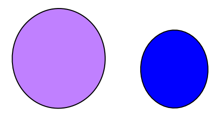
Matrix No. 5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



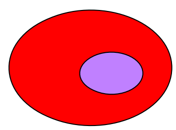
Matrix No. 6

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



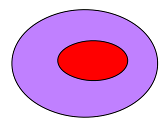
Matrix No. 7

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



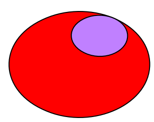
Matrix No. 8

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



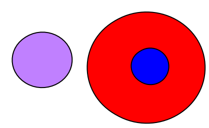
Matrix No. 9

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



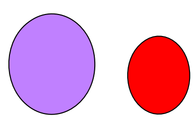
Matrix No. 10

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



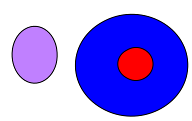
Matrix No. 11

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



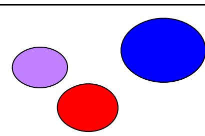
Matrix No. 12

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



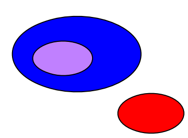
Matrix No. 13

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



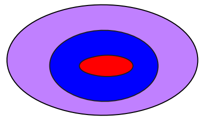
Matrix No. 14

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



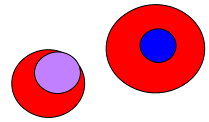
Matrix No. 15

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



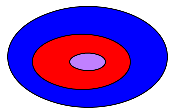
Matrix No. 16

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



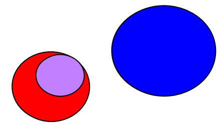
Matrix No. 17

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



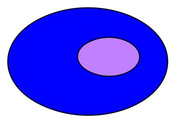
Matrix No. 18

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



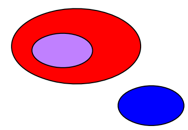
Matrix No. 19

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



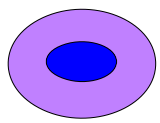
Matrix No. 20

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



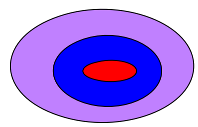
Matrix No. 21

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



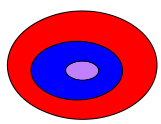
Matrix No. 22

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix No. 23

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



Matrix No. 24

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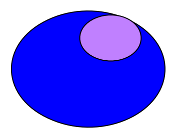
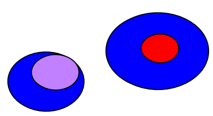


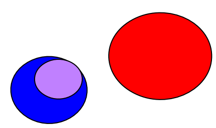
Matrix No. 25

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



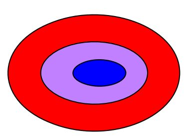
Matrix No. 26

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



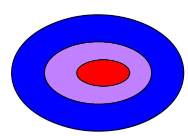
Matrix No. 27

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



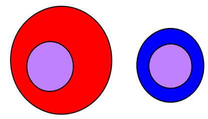
Matrix No. 28

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



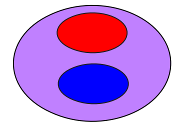
Matrix No. 29

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



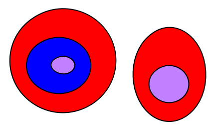
Matrix No. 30

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



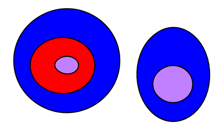
Matrix No. 31

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



Matrix No. 32

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Matrix No. 33

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

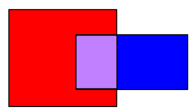


Table 3. The 33 topological relationships for complex regions.

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