COT 5405 : ANALYSIS OF ALGORITHMS
Midterm Solution 2

Q2:

Solution:

The potential function is the sum of the tree heights $\phi(Di) = \sum_{j=1}^{size(Di)} log(j)$

proof :

1. $\phi(D_0) = 0$
2. $\phi(Di) \geq 0$ for all i
3. INSERT:
   - The amortized cost is $\hat{c_i} = c_i + \phi(D_i) - \phi(D_{i-1})$
   - Since INSERT operation is $O(log(size(D_i))$ worst-case time, and insertion will increase one node, so $size(D_i) = size(D_{i-1}) + 1$
   - so $\hat{c_i} = c_i + \phi(D_i) - \phi(D_{i-1}) \leq log(size(D_i)) + \sum_{j=1}^{size(D_i)} log(j) - \sum_{j=1}^{size(D_{i-1})} log(j) = log(size(D_i)) + \sum_{j=1}^{size(D_i)} log(j) - \sum_{j=1}^{size(D_i)-1} log(j) = log(size(D_i)) + log(size(D_i)) \leq O(log(n))$
4. EXTRACT-MAX
   - The amortized cost is $\hat{c_i} = c_i + \phi(D_i) - \phi(D_{i-1})$
   - Since EXTRACT-MAX operation need to remove max and heapify to maintain max-heap, which takes $O(d + log(size(D_i)))$ worst-case time, and removing max will decrease one node, so $size(D_i) = size(D_{i-1}) - 1$ (d is constant)
   - so $\hat{c_i} = c_i + \phi(D_i) - \phi(D_{i-1}) \leq d + log(size(D_i)) + \sum_{j=1}^{size(Di)} log(j) - \sum_{j=1}^{size(D_{i-1})} log(j) = d + log(size(D_i)) + \sum_{j=1}^{size(Di)} log(j) - \sum_{j=1}^{size(D_i)+1} log(j) \leq d + log(size(D_i)) - log(size(D_i)) \leq O(1)$

Grading policy:

1. Choose correct potential function 5 points
2. verify Insert operation 7 points
3. verify Extract-MAX operation 8 points

Q3:

Solution:

a) Claim 1: True

proof:

Let T be the maximum spanning tree of G, T' be the minimum spanning tree of G'

Assume the contrary, The maximum spanning tree T of $G = (V, E)$ with one-to-one weight function w is not the same as a minimum spanning tree T' of $G' = (V, E)$with weight function $w' = -w$. There is at least one edge (u, v)in T which is not in T'.

Let (S,V-S) be any cut of G such that u and v on separate sides.

Also , (S, V-S) will be a cut of G'.

since, edge (u,v) is not in T', there exist a edge (x, y) be the light edge crossing (S,V-S) which is in T'.

1. if $w(u, v) = w(x, y)$ then $T'_{min} = \{T' - (x, y)\} \cup (u, v)$ is still a minimum spanning tree. Continue finding another different edge pair with same weight, we can iterate change the same weight edge until $T = T'_{min}$ which is contradiction

2. if $w(u, v) \neq w(x, y)$

(x,y) is not in T because the edge (x,y) forms a cycle with the edges on the path from x to u, u to v, v to y in T(suppose x,u are on the same side, v,y on the other side).

Let T_{max} be a new spanning tree such that $T_{max} = \{T - (u, v)\} \cup (x, y)$

since (x,y) is the light edge crossing (S,V-S) so $w'(x, y) \leq w'(u, v)$

therefore $w(T_{max}) = w(T) - w(u, v) + w(x, y) = w(T) + w'(u, v) - w'(x, y) \geq w(T)$

But T is the maximum spanning tree, so that $w(T) \geq w(T_{max})$ which is the contradiction.

b) Claim2:True

proof:

Assume the contrary, the edge e(u,v) of minimum spanning tree T of G is not the lightest edge across cut (S,V-S) of G.

Let (x,y) be the lighest edge across (S,V-S) of G.

(x,y) is not in T otherwise the ege (x,y) forms a cycle with the edges on the path from x to u, u to v, v to y in T(suppose x,u are on the same side, v,y on the other side).

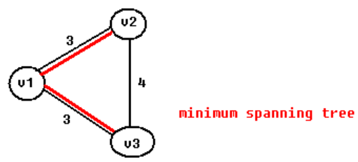
Let T_{min} be a new spanning tree such that $T_{min} = \{T - (u, v)\} \cup (x, y)$

since (x,y) is the lightest edge crossing (S,V-S) so $w'(x, y) < w'(u, v)$

therefore $w(T_{min}) = w(T) - w(u, v) + w(x, y) < w(T)$

But T is the minimum spanning tree, so that is the contradiction.

c) False;


v2
3
4
v1
3
v3
minimum spanning tree