Numerical Analysis
COT4501
Spring 2005
Midterm I

24th February 2005

1. **[25 points overall]**

[10 points] What is the second order Taylor series approximation $p_2(x)$ and the remainder $R_2(x)$ for $f(x) = x^m \log x$ for an integer $m > 2$ when the approximation is carried out at $x_0 = 0$ and in the interval $[0, 1]$. What happens if $m = 2$?

[15 points] What is the first order Taylor series approximation $p_1(x)$ and the remainder $R_1(x)$ of $f(x) = x^{\frac{1}{x}}$ when the approximation is carried out at $x_0 = 1$ and in the interval $[0.5, 2]$? [Hint: If you have trouble differentiating $x^{\frac{1}{x}}$, consider transforming $x^{\frac{1}{x}}$ into the form $e^{f(x)}$ which can be easily differentiated.]

2. **[30 points overall]**

[10 points] Consider the following ordinary differential equation (ODE)

$$y' = e^{t-y}.$$

If $y(t_0) = -1$, what is the Euler approximation for the ODE? Compute y_1 for $t_0 = 0$ and $h = \frac{1}{4}$. If the solution $y(t) = \log(e^t - 1 + e^{-1})$, then what is the error at $t_1 = h$ where h is now arbitrary? You do not need to explicitly calculate anything.

[20 points overall] Extension of Euler's method to partial differential equations: Consider the following partial differential equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

where t is time and x is space (one-dimensional).

[10 points] Write down an Euler-like approximation y_{t_n,x_m} of the above partial differential equation for $t_n > 0$ and $x_m > 0$ and where (n, m) are integers. Clearly state all the initial conditions you will require in order to compute the Euler approximation. [Hint. You should consider approximating both the derivative w.r.t. time and the derivative w.r.t. space using derivative approximations.]

[10 points] What is the error of the approximation $|y(t_n, x_m) - y_{t_n,x_m}|$? That is, clearly write down all remainder terms emanating from both derivative approximations. You do not need to bound the error.

3. **[15 points overall]**

Extend the linear interpolation formula to quadratic interpolation. Given three points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$:

1. **[10 points]** Construct a second order polynomial approximation $p_2(x)$ of the form $p_2(x) = ax^2 + bx + c$ such that $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$.

2. **[5 points]** What are the values of a, b, and c?

4. **[10 points]**

If one root of the equation $f(x) = 0$ occurs at $x = 0$, give sufficient conditions that the functions $f(x)$ and $f'(x)$ have to obey in order for Newton's method to never diverge to $\pm\infty$ regardless of the initial condition $x_0 > 0$ or $x_0 < 0$. [Note: This is different from requiring that Newton's method converges since it could oscillate or show chaotic behavior.]

List of Useful Formulae

Taylor series approximation: $f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^i f^{(i)}(x_0)}{i!} + \frac{(x-x_0)^{(n+1)} f^{(n+1)}(\xi_{[x_0,x]})}{(n+1)!}$ with $\xi_{[x_0,x]}$ in the interval $[x_0, x]$.

Euler's method: Approximation of $y' = f(t, y)$ is $y_{n+1} = y_n + hf(t_n, y_n)$ with $y_0 = y(t_0)$.

Difference Approximations of Derivatives: $f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{1}{2}hf''(\xi_{x,h})$, $f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} - \frac{1}{12}h^2 f''''(\xi_{x,h})$.

Linear Interpolation formula: $p_1(x) = \frac{x_1-x}{x_1-x_0}f(x_0) + \frac{x-x_0}{x_1-x_0}f(x_1)$.

Bisection method: $|\alpha - x_n| \leq \left(\frac{1}{2}\right)^n (b-a)$ with the midpoint of the interval chosen at each step.

Newton's method: Solution of $f(x) = 0$ is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with initial condition x_0.

Quadratic formula: Solution of $ax^2 + bx + c = 0$ is $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.