

Section 1.2

4. From Taylor's Theorem expanding at $x_0 = 0$,
since, $\cos \xi_x$ is bounded by 1, so
 $|\sin x - x| \leq \frac{1}{3!}x^3$. And so we can write: $\sin x = x + \mathcal{O}(x^3)$

5.

Again, from Taylor's theorem expanding at $x_0 = 0$ we have,

$$(1+x)^{-1} = 1 - x + x^2 - \frac{x^3}{(1+\xi_x)^4}$$

Because, $0 \leq \xi_x \leq x$, Therefore the last term is maximized at $\xi_x = 0$.

So, $|(1+x)^{-1} - (1-x+x^2)| \leq x^3$. Hence, $\mathcal{O}(x^3)$.

7.

From the result in Q6 we have,

$\sum_{k=0}^{k=\infty} r^k = \sum_{k=0}^{k=n} r^k + \mathcal{O}(r^{n+1})$ because $\sum_{k=0}^{k=\infty} r^k = \frac{1}{1-r}$ and this is true only for $|r| < 1$ and not otherwise.

So, now for Q7, $r = \frac{1}{e}$. Putting $n = 8$ we have the error on the order of $e^{-9} = 1.2341 \times 10^{-4}$.

Putting $n = 9$ the error is $e^{-10} = 4.5400 \times 10^{-5}$.

So, number of terms = 10 and not 9.

14.

The definition of Big Oh requires a constant 'C' to exist that works for all 'h'. Hence 'C' cannot depend on 'h'.

Section 1.3

11.

The second approach is more accurate because in the first approach subtractive cancellation would destroy almost all the accuracy in the calculation when x is very small.

The following matlab script shows the cancellation:

Note the cancellation when $x = 0.9999999$ i.e., when $x \approx 1$ but not exactly equal to 1.

```

x = 0.9999999
f1 = 1 - x + (x^2)/2 - (x^3)/6 ;
f2 = 1/(1 + x + (x^2)/2 + (x^3)/6) ;
val = exp(-x) ;
err1 = abs(val-f1) % err1 is 0.0345
err2 = abs(val-f2) % err2 is 0.0071

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Section 1.5

1.

In the formula (1.11) on page 32 plug in $k = 3$ and $\xi = 0$ for maximum $E_k(x)$. We want $E_k(x) \leq 10^{-3}$.

The range of values of x that we get is $x \in [0, 10^{-3} * \sqrt{\pi} * 108]$ or $x \in [0, 0.8322]$

5.

The $E_k(x)$ term now is:

$\frac{2(-1)^{k+1}}{(k+1)!\sqrt{\pi}} \int_0^x t^{2k+2-p} e^c dt$ where c depends upon t and $-t^2 \leq c \leq 0$. Using Integral Mean Value theorem because both the integrands are positive we have

$$E_k(x) = \frac{2(-1)^{k+1}}{(k+1)!\sqrt{\pi}} \frac{e^{\xi} x^{2k+3-p}}{(2k+3-p)}$$

Take $p = 1$, $\xi = 0$, and $x = 0.5$ we get $k = 2$.

7.

Given $f(x) = \int_0^x t^{-1} \sin(t) dt$, $x \in [-\pi/4, \pi/4]$. The Taylor expansion for $g(t) = \sin(t)$ at $t_0 = 0$ is

$$\begin{aligned} \sin(t) &= p_n(t) + R_n(t), \text{ where } |R_n(t)| = \left| \frac{t^{n+1}}{(n+1)!} g^{(n+1)}(\xi) \right|, \xi \in [0, t] \\ f(x) &= \int_0^x t^{-1} (p_n(t) + R_n(t)) dt \\ \Rightarrow E(x) &= \left| f(x) - \int_0^x t^{-1} p_n(t) dt \right| = \left| \int_0^x t^{-1} R_n(t) dt \right| = \frac{1}{(n+1)!} \left| \int_0^x t^{-1} (t^{n+1}) g^{(n+1)}(\xi) dt \right|, \xi \in [0, t] \\ &\leq \frac{g^{(n+1)}(\xi)}{(n+1)!} \left| \int_0^x t^n dt \right|, \xi \in [0, x] \text{ (due to IMVT) (since, } g^{(n+1)}(t) \leq 1) \\ &\leq \frac{1}{(n+1)!} \left| \frac{x^{n+1}}{n+1} \right| \leq \frac{1}{(n+1)!} \frac{(\pi/4)^{n+1}}{n+1} \text{ (after substituting } x = \pi/4) \\ &\leq 10^{-4} \Rightarrow n \geq 5 \end{aligned}$$

$$\Rightarrow f(x) \approx \int_0^x t^{-1} \left(t - \frac{t^3}{3! \cdot 3} + \frac{t^5}{5!} \right) dt = x - \frac{x^3}{3! \cdot 3} + \frac{x^5}{5! \cdot 5}$$

9.

1. Given $f(x) = \sqrt{x}$, $x \in [1/4, 1]$ and $x_0 = 9/16$, error in degree-4 Taylor polynomial is

$$\begin{aligned} R_4(x) &= \frac{(x - x_0)^5}{5!} f^{(5)}(\xi), \quad \xi \in [x_0, x] \\ |R_4(x)| &= \left| \frac{(x - x_0)^5}{5!} \left(\frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2^5} \right) \xi^{-9/2} \right|, \quad (\text{with } \xi = 1/4, x = 1, x_0 = 9/16) \\ &\leq |(7/16)^5 \cdot 7 \cdot 2| \approx 0.2244 \end{aligned}$$

11.

The answer for Q10 given at the back of the book is actually for Q11.