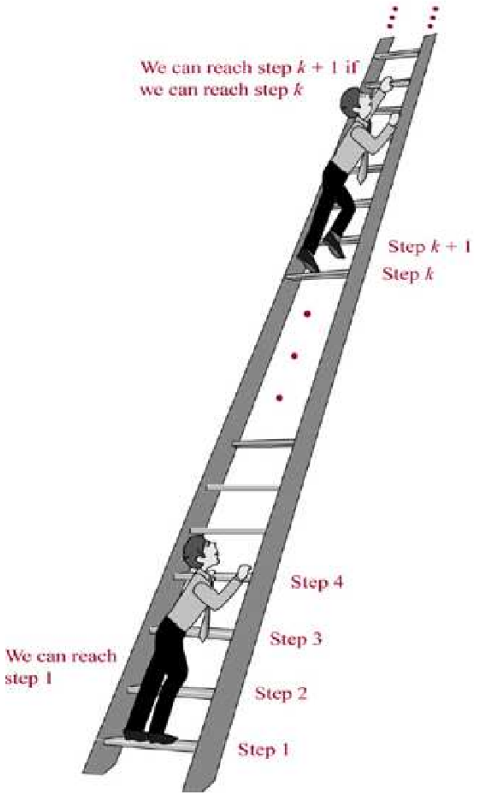
COT3100: Strong Induction (Sec 4.2)



If you know how to

- Reach the first step of a ladder
- Reach step $k+1$, if we can reach all the first k steps

Then, you can reach any step of the ladder.

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1 – Strong Induction

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION.

To prove that $P(n)$ is true $\forall n \in \mathbb{Z}^+$, where $P(n)$ is a predicate, we complete two steps:

BASIS STEP. We show that P(1) is true.

INDUCTIVE STEP.

We show that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true $\forall k \in \mathbb{Z}^+$

The assumption that $P(j)$ is true $\forall j = 1, 2, \ldots, k$ is called the

inductive hypothesis

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2 – Example: Product of Primes

THEOREM. **Any integer greater than 1 can be written as the product of primes.**

Prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$

$2 = 2$
$3 = 3$
$4 = 2 \cdot 2$
$5 = 5$
$6 = 2 \cdot 3$
$7 = 7$
$8 = 2 \cdot 2 \cdot 2$
$9 = 3 \cdot 3$
$10 = 2 \cdot 5$
$11 = 11$
$12 = 2 \cdot 2 \cdot 3$
$13 = 13$
$14 = 2 \cdot 7$
$15 = 3 \cdot 5$
$16 = 2 \cdot 2 \cdot 2 \cdot 2$
$17 = 17$
$18 = 2 \cdot 3 \cdot 3$
$19 = 19$

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THEOREM. **Any integer greater than 1 can be written as the product of primes.**

$P(n)$: n can be written as the product of primes.

BASIS STEP. $P(2)$ is true, because 2 is a prime number.

INDUCTIVE STEP. We want to prove $[P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

Assuming all $P(j)$ is true $\forall j(2 \leq j \leq k)$, prove $P(k+1)$ is true.

$k+1$ is either prime or composite.

CASE 1. If $k+1$ is prime, proof is complete because $P(k+1)$ is true.

CASE 2. If $k+1$ is composite then $\exists a, b$ s.t. $k+1 = ab$ and

$$2 \leq a \leq b < k+1.$$

By inductive hypothesis a and b can be written as product of primes.

Hence $k+1 = ab$ can be written as product of primes.

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3 – Example: Postage (Strong Induction)

THEOREM. **Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.**

PROOF BY STRONG INDUCTION.

BASIS STEP. $P(12)$ is true, because $12 = 4 + 4 + 4$.
$P(13)$ is true, because $13 = 5 + 4 + 4$.
$P(14)$ is true, because $14 = 5 + 5 + 4$.
$P(15)$ is true, because $15 = 5 + 5 + 5$.

INDUCTIVE STEP.

HYPOTHESIS. $P(k-3)$, $P(k-2)$, $P(k-1)$, and $P(k)$ is true.

We can form a postage of $k+1$ cents, using the postage for $k-3$ cents and a 4-cent stamp. Q.E.D.

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4 – Example: Postage (Mathematical Induction)

THEOREM. **Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.**

BASIS STEP. $P(12)$ is true, because $12 = 4 + 4 + 4$.

INDUCTIVE STEP.

HYPOTHESIS. $P(k)$ is true.

CASE 1. At least one 4-cent stamp is used for postage of k cents.

Replace one 4-cent stamp with one 5-cent stamp.

CASE 2. No 4-cent stamp is used for postage of k cents.

Then, at least three 5-cent stamp is used for postage of k cents.

Replace three 5-cent stamp with four 4-cent stamp. Q.E.D.

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Replace three 5-cent stamp with four 4-cent stamp. Q.E.D.