1 – Proof by Contradiction-Example

EXAMPLE. Prove that "if $3n + 2$ is odd, then n is odd."

Equivalent to "Either $3n + 2$ is not odd or n is odd."

Suppose for the sake of contradiction "$3n + 2$ is odd and n is not odd."

n is not odd	$\rightarrow$	n is even	(n is either odd or even but not both)
	$\rightarrow$	$n = 2k$	(n is divisible by 2)
	$\rightarrow$	$3n + 2 = 3(2k) + 2$	(substitute 2k for n)
	$\rightarrow$	$3n + 2 = 2(3k + 1)$	(simple algebra)
	$\rightarrow$	$3n + 2$ is even	(divisible by 2)
	$\rightarrow$	$3n + 2$ is not odd	THIS IS A CONTRADICTION

Hence "Either $3n + 2$ is not odd or n is odd."

2 – Proof by Cases

$$p \rightarrow q \quad \equiv p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q \qquad \text{(Break the premise)}$$

$$\equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q) \qquad \text{(Then prove each case)}$$

EXAMPLE. Prove that "if n is an integer, then $n^2 \geq n$."

Three cases: n is 0, or a positive integer or a negative integer.

Case 1: When $n = 0$, since $0^2 = 0$, the conlusion is correct.

Case 2: When $n \geq 1$, multiply both sides with n and get $n^2 \geq n$.

Case 3: When $n \leq -1$. Note that $n^2 > 0$. Hence, $n^2 \geq n$.

MAKE SURE TO COVER ALL POSSIBLE CASES.

3 – Disproving via Counterexamples

To prove a theorem is false, we can construct counterexamples.

EXAMPLE. Prove or disprove that ”every positive integer is the sum of square of two integers.”

3 is a counterexample.

The only perfect squares not exceeding 3 are 0 and 1.

3 cannot be written as a sum of two terms each of which is 0 or 1.

4 – Equivalence Proofs

EXAMPLE. Show that "$\forall x$, x is even if and only if x^2 is even."

"$\forall x$, if x is even then x^2 is even, and if x^2 is even then x is even."

PROOF STRUCTURE.

Case 1. (only if part, necessity)

x is even $\longrightarrow$ x^2 is even

Case 2. (if part, sufficiency)

x^2 is even $\longrightarrow$ x is even

x is not even $\longrightarrow$ x^2 is not even (contrapositive)

5 – Equivalance Proofs (necessity)

EXAMPLE. Show that "$\forall x$, x is even if and only if x^2 is even."

PROOF.

Case 1. (only if part, necessity)

x is even	$\longrightarrow$	$x = 2k$	(x is divisible by 2)
	$\longrightarrow$	$x^2 = 4k^2$	(square both sides)
	$\longrightarrow$	$x^2 = 2(2k^2)$	(simple algebra)
	$\longrightarrow$	x^2 is even	(x^2 divisible by 2)

6 – Equivalance Proofs (sufficiency)

EXAMPLE. Show that "$\forall x$, x is even if and only if x^2 is even."

PROOF (CONT.)

Case 2. (if part, sufficiency)

x is not even	$\rightarrow$	x is odd	(x is even or odd but not both)
	$\rightarrow$	$x = 2k + 1$	(by definition of odd number)
	$\rightarrow$	$x^2 = 4k^2 + 4k + 1$	(square both sides)
	$\rightarrow$	$x^2 = 2(2k^2 + 2k) + 1$	(factor the first two terms by 2)
	$\rightarrow$	x^2 is odd	(by definition of odd number)
	$\rightarrow$	x^2 is not even	

7 – Equivalance Proofs (Several Propositions)

Show that $p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$

Use the Equivalance

$$p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n \equiv (p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge \ldots \wedge (p_n \rightarrow p_1)$$

EXAMPLE. Show that the following statements about integer n are equivalent.

p_1 : n is even

p_2 : $n - 1$ is odd

p_3 : n^2 is even

PROOF. Show $p_1 \rightarrow p_2$

Show $p_2 \rightarrow p_3$

Show $p_3 \rightarrow p_1$

PROVE EACH IMPLICATION AS AN EXERCISE.

8 – Existence Proofs

We want to prove $\exists x P(x)$, where $P(x)$ is a predicate.

Constructive proof for existence:

Find an element a of the domain such that $P(a)$ is TRUE.

Then $\exists x P(x)$ is TRUE by Existential Generalization.

Nonconstructive proof for existence:

Use Proof by contradiction.

That is Negation of $\exists x P(x)$, which is $\forall x \neg P(x)$ implies a contradiction.

Prove the existence without giving a specific example.

9 – Constructive Proof - Example

We want to prove $\exists x P(x)$, where $P(x)$ is a predicate.

Constructive proof for existence:

Find an element a of the domain such that $P(a)$ is TRUE.
Then $\exists x P(x)$ is TRUE by Existential Generalization.

THEOREM. There exists an integer solution to the equation $x^2 + y^2 = z^2$.

PROOF. Choose $x = 3$, $y = 4$, and $z = 5$.

10 – NonConstructive Proof - Example

We want to prove $\exists x P(x)$, where $P(x)$ is a predicate.

Nonconstructive proof for existence:

Prove the existence without giving a specific example.

THEOREM. There exists irrational numbers x and y such that x^y is rational.

PROOF. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational then we are done.

Otherwise, consider the number $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

This is number is rational because $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^2 = 2$.

Either $x = \sqrt{2}$ and $y = \sqrt{2}$ or $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$.

But we do not know which pair has the desired poperty.

11 – Uniqueness Proofs

Existence of a unique element with a particular property.

Uniqueness proofs are two parts:

Existence: Show that an element x with the desired property exists.

Uniqueness: Show that if x and y both have the desired property, then $x = y$.

Existence + Uniqueness: $\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$

12 – Uniqueness Proofs - Example

Uniqueness proofs are two parts:

Existence: Show that an element x with the desired property exists.

Uniqueness: Show that if x and y both have the desired property, then $x = y$.

EXAMPLE. Show that if $a \neq 0$ and b are real numbers, then there is a unique real number r such that $ar + b = 0$.

PROOF. Existence: Such r always exists, namely $r = -b/a$.

Uniqueness: Suppose s is a real number such that $as + b = 0$.

Then, $as + b = ar + b$.

Then, $as = ar$.

Hence, $s = r$.