**COT3100**
**Quiz 4   April 3, 2009**
**Name:**

**Write your name and solutions clearly on this page (and back if necessary). There are 10 points for this quiz.**

**Problem 1. (6 pts)**
Prove using the principle of mathematical induction that $n! < n^n$ for all $n > 1$. (In the proof, you have to state the inductive hypothesis clearly.)

**Solution:** The base case is easy to check: $n = 2$, we clearly have $2 = 2! < 4 = 2^2$.
The inductive hypothesis: $n! < n^n$ is true for $n \leq K$, for $K \geq 2$.
Now we need to show that $(K+1)! < (K+1)^{K+1}$: from $K! < K^K$, we have

$$(K+1) \cdot K! < (K+1) \cdot K^K.$$

The left hand side is $(K+1)!$ and in the right hand side, we have

$$K^K < (K+1)^K.$$

Putting these together, we have

$$(K+1)! = (K+1) \cdot K^K < (K+1) \cdot (K+1)^K = (K+1)^{K+1}.$$

**Problem 2. (4 pts)**
**(a)** (1 pt) Give a recursive definition for the Fibonacci numbers.
**(b)** (3 pts) Let $f_i$ denote the $i$th Fibonacci number. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.
**Solution:** Fibonacci numbers $\{f_0, f_1, \cdots, f_n, ..\}$ are defined recursively as follows: $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Part (b) can be proved using induction on $n$. The base case is $n = 1$, and this follows directly from the fact that $f_1 = f_2 = 1$. Now assume that the equation works for all $n \leq K$: $f_1 + f_3 + \cdots + f_{2K-1} = f_{2K}$. We need to show that the equation is true for $n = K + 1$. This is easy: From the definition

$$f_{2(K+1)} = f_{2(K+1)-1} + f_{2K}.$$

Using inductive hypothesis to expand $f_{2K}$ in the equation above, we have

$$f_{2(K+1)} = f_{2K+1} + f_{2K-1} + f_{2K-3} + \cdots + f_3 + f_1.$$

The completes the induction.