COT3100: Discrete Structures – Spring 2007

Final Exam (Sample)

This is a closed book exam. No collaborations are allowed. Your solutions should be concise, complete, and legible. Use only the space provided below each question. **Good luck!**

Your name: _____

	Credit	Max
Problem 1		10
Problem 2		10
Problem 3		15
Problem 4		10
Problem 5		15
Problem 6		10
Problem 7		10
Problem 8		20
Total		100

1. [10 pts] A simple graph is called k-**regular** if every vertex of this graph has degree k. Show that if a bipartite graph $G = (V, E)$ is k-regular for some positive integer k and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$. That is, show that the two sets in a bipartition of the vertex set of an k-regular graph must contain the same number of vertices.

2. [10 pts] Schedule the practice hours (one hour per week for each team) for the Gainesville High School's Basketball, Football, Soccer, Baseball, and Volleyball teams using the fewest number of time slots per week, if there are **no** student athletes playing both Basketball and Baseball, both Football and Soccer, both Volleyball and Basketball, both Baseball and Volleyball, but there are athletes playing in every other combination of sports.
(*Go Purple Hurricanes!*).

3. [15 pts] In an old puzzle, a farmer needs to carry a wolf, a goat, and a cabbage across a river. The farmer only has a small boat, which can carry the farmer and only one object (an animal or a vegetable). He can cross the river repeatedly. However, if the farmer is on the other shore, the wolf will eat the goat and the goat will eat the cabbage. We can describe each state by listing what is on each shore. For example, we can use the pair $(\mathrm{FG}, W\mathrm{C})$ for the state where the farmer and goat are on the first shore and the wolf and the cabbage are on the other shore. [The symbol $\emptyset$ is used when nothing is on a shore, so that $(\mathrm{F}W\mathrm{GC}, \emptyset)$ is the initial state.]

   (a) Find all allowable states of the puzzle, where neither the wolf and the goat nor the goat and the cabbage are left on the same shore without the farmer. Construct a graph such that each vertex of this graph represents an allowable state and the vertices representing two allowable states are connected if it is possible to move from one state to the other using one trip of the boat.

(b) Note that finding a path from the vertex representing $(\mathrm{FWGC}, \emptyset)$ to the vertex representing $(\emptyset, \mathrm{FWGC})$ solves the puzzle. Find two different solutions of the puzzle, each using seven crossings.

(c) Suppose that the farmer must pay a toll of one dollar whenever he crosses the river with an animal. Which solution of the puzzle should the farmer use to pay the least total bill?

4. [10 pts] Prove that $(x+1)!$ is **not** $O(x!)$ using only the definition of Big-Oh notation. *(Hint:* $\log(a \cdot b) = \log a + \log b$*)*

5. [15 pts] Use induction to prove that $(a - b)$ divides $a^n - b^n$ for all real numbers a and b and integer $n \geq 1$.

6. [10 pts] Given a set A with n elements. Find the number of symmetric binary relations on the set A.

7. [10 pts] You have 40 different books (20 math books, 15 history books, 5 geography books). You pick two books at random, one at a time. What is the probability that the two books are from different disciplines?

8. [20 pts] In the questions (a) to (d) fill in the blanks
(No need to explain your answer.)

(a) $K_{m,n}$ has _____ edges and _____ vertices.

(b) The length of the longest simple circuit (cycle) in K_5 is _____.

(c) Give a recurrence relation for the number of edges, e_n of the n-cube graph, Q_n.

$e_n =$ _____

(d) There are _____ 1s and _____ 0s in the adjacency matrix of C_n.

(e) Determine if each of the following statements is TRUE or FALSE. (No need to explain your answer.)

- $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology.

- Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be function defined as $f(n) = 3n^2 - 1$. f is a one-to-one function.