Logic and Proofs

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1 – Translating from Natural Languages

EXAMPLE. Translate the following sentence into a logical expression:

"I met my ex-girlfriend today and either she grew taller or I got shorter."

p = "I met my ex-girlfriend today."

q = "She grew taller."

r = "I got shorter."

$$p \wedge (q \vee r)$$

$p \wedge q \vee r$ would mean something else:

"I got shorter or I met my ex-girlfriend today and she grew taller"

$p \wedge (q \oplus r)$ would also mean something else...

EXAMPLE. Translate the following sentence into a logical expression:

"You get an A only if you score at least 50% on the midterm or you submit a HW."

p = "You get an A."

q = "You score at least 50% on the midterm."

r = "You submit a HW."

$$p \longrightarrow (q \vee r)$$

$(q \vee r) \longrightarrow p$ would mean something else:

"You get an A if you score at least 50% on the midterm or you submit a HW."

2 – Solving Puzzles using Propositional Logic

EXAMPLE. An island has two types of inhabitants: knights, who always tell the truth, and knaves, who always lie. You meet two people from this island: Alice and Bob. Alice says "Bob is a knight" and Bob says "The two of us are opposite types."

Let p= "Alice is a knight." $\neg p$= "Alice is a knave."

Let q= "Bob is a knight." $\neg q$= "Bob is a knave."

Alice said q and Bob said $p \oplus q$.

Island Rule: Alice is a knight iff what she said is true and Bob is a knight iff what he said is true. $(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$

Let p= “Alice is a knight.” $\neg p$= “Alice is a knave.”

Let q= “Bob is a knight.” $\neg q$= “Bob is a knave.”

Alice said q and Bob said $p \oplus q$.

Island Rule: Alice is a knight iff what she said is true and Bob is a knight iff what he said is true. $(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$

	Alice said			Bob said	Island rule
p	q	$\neg p$	$\neg q$	$p \oplus q$	$(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	F	T

Hence, Both Alice and Bob are knaves.

3 – Tautology and Contradiction

A compound statetement that is always true is called a *tautology*.

A compound statetement that is always false is called a *contradiction*.

A compound proposition that is neither a tautology nor a contradiction is called a *contingency*

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

4 – Equivalence

The compound propositions p and q are called *logically equivalent*, denoted by $p \equiv q$, if $p \leftrightarrow q$ is a tautology.

Remark: The symbol $\equiv$ is not a logical connective and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

Use truth tables to determine logical equivalences. Two compound propositions are equivalent if their columns in the truth table are the same.

5 – Proving De Morgan Laws

TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

EXAMPLE. State the negation of “I am a doctor or a lawyer.”

“I am not a doctor and I am not a lawyer.”

TABLE 2 De Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

EXAMPLE. State the negation of "She is rich and beautiful."

"She is either not rich or not beautiful."

6 – Equivalence Example

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

EXAMPLE. “If you are a freshman you can get a free ticket”

“Either you are not a freshman or you can get a free ticket.”

7 – Distributive Law

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$
$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

TABLE 5 A Demonstration That $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ Are Logically Equivalent.

p	q	r	$q \land r$	$p \lor (q \land r)$	$p \lor q$	$p \lor r$	$(p \lor q) \land (p \lor r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws

COT3100: Propositional Equivalences

Equivalence	Name
$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation Laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg (p \rightarrow \neg q)$$

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

TABLE 8 **Logical Equivalences Involving Biconditionals.**

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

8 – Constructing new equivalences

EXAMPLE. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{(See page 10 of these slides)} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{(De Morgan's Law)} \\
&\equiv p \wedge \neg q && \text{(Double negation)}
\end{aligned}
$$

9 – Proving a Tautology

EXAMPLE. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$(p \wedge q) \rightarrow (p \vee q)$	$\equiv$	$\neg(p \wedge q) \vee (p \vee q)$	(See page 10 of these slides)
	$\equiv$	$(\neg p \vee \neg q) \vee (p \vee q)$	(De Morgan's Law)
	$\equiv$	$(\neg p \vee \neg q \vee p) \vee q$	(Associative Law)
	$\equiv$	$(\neg p \vee p \vee \neg q) \vee q$	(Commutative Law)
	$\equiv$	$(\neg p \vee p) \vee (\neg q \vee q)$	(Associative Law)
	$\equiv$	$T \vee T$	(Negation Law)
	$\equiv$	T	