Discrete Quadratic Curvature Energies

Motivation: thin shell (surface) *bending and stretching* as in cloth dynamics (also think skin, paper, etc.); also "fairing" (Willmore energy)
Dynamics: For a surface at position $\mathbf{x}(u,v,t) \in \mathbb{R}^3$, moving in a velocity field $\mathbf{v}(u,v,t) \in \mathbb{R}^3$, a mass matrix $M \in \mathbb{R}^{3\times 3}$, the Hessian with respect to $\mathbf{x}$ denoted by Hess and the 'energies' $E_b, E_p \in \mathbb{R}$, Newton's formula for moving the surface S with position $\mathbf{x}$ subject to the forces $\mathbf{F}_E$ and $\mathbf{F}_D$ is

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & (\rho M)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{F}_E(\mathbf{x}(t)) + \mathbf{F}_D(\mathbf{x}(t)) \end{bmatrix} \tag{1}$$

$$\mathbf{F}_D := -(\alpha_1 M + \alpha_2 \mathrm{Hess}(E_b) + \alpha_2 \mathrm{Hess}(E_p))\mathbf{v} \tag{2}$$

$$\mathbf{F}_E := -\nabla E_b - \nabla E_p. \tag{3}$$

Here the suscript D stands for damping, E for energy; and b for bending and p for stretching. The decisive ingredient in the motion of S is the definition of the bending energy E_b. The equations may be solved by (semi-)implicit Euler steps.
Energies: The stretching energy E_p penalizes stretching of the surface. This is captured by looking at the first fundamental form[1] $\mathbb{I}$. Since E_p is *assumed* in the model to be at least two orders of magnitude larger than E_b, we may assume that the deformation minimizes E_p and is therefore almost *isometric* ($\mathbb{I}$ does not change).

We choose to model bending as an integral of *mean crvature* H (rather than Gauss curvature K, whose integral is determined by the Gauss-Bonnet formula or total curvature $\kappa_1^2 + \kappa_2^2$) where κ_1 and κ_2 are the principle curvatures

$$E_b := \frac{1}{2}\int_S H^2 dA, \quad H := \kappa_1 + \kappa_2, \tag{4}$$

$$= \frac{1}{2}\int_S \langle \Delta\mathbf{x}, \Delta\mathbf{x}\rangle_{\mathbb{R}^3} dA \tag{5}$$

Here $\Delta\mathbf{x}$ is the Laplace-Beltrami operator, i.e. $\Delta = \nabla \cdot \nabla$ where the differentiation (gradient) is on the surface (intrinsic) rather than in the (ambient) space R^3. If $\mathbb{I}$ is constant, E_b is quadratic in $\mathbf{x}$.
Discretization: The solution of the *Dirichlet Problem*

$$\Delta x = 0, x_{\partial S} = g$$

(where ∂S is the boundary of S and x could be one coordinate of $\mathbf{x}$) is a minimizer of the *Dirchlet energy*

$$D(x) := \int_S \langle \nabla x, \nabla x \rangle dA \tag{6}$$

D is (D1) nonegative, (D2) zero iff x=const, (D3) scale-invariant if x is bivariate. (The area A scales by λ^2 and ∇ by λ^{-1} if x is scaled by $\lambda \in \mathbb{R}$.)

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[1] web-search notions you do not know

We discretize D and x with k samples in each parameter by the quadratic form $L \in \mathbb{R}^{k\times k}$ (for Laplace) as

$$D(x) \approx d(\mathbf{u},\mathbf{u}) := \mathbf{u}^{\mathtt{t}} L\mathbf{u}. \tag{7}$$

where $\mathbf{u} \in \mathbb{R}^k$ is the vector of samples of x. To match (D1,D2,D3) L should be (L1) symmetric positive semidefinite, (L2) vanish exactly when $\mathbf{u}$ is constant and be (L3) invariant under uniform scaling. (note that $d(\mathbf{u}+c,\mathbf{u}+c) = d(\mathbf{u},\mathbf{u}) + d(c,c) + 2d(\mathbf{u},c)$ and hence for $d(c,c) = 0$ (L2), $d(\mathbf{u},c) = 0$ must hold since otherwise $d(\mathbf{u}+c,\mathbf{u}+c) = d(\mathbf{u},\mathbf{u}) + ad(\mathbf{u},c) < 0$ for suitably chosen $a \in \mathbb{R}$, contradicting nonegativity of d.) We add (L4): $L\mathbf{x} = 0$ if $\mathbf{x}$ is part of a plane.

The solution of the *Poisson Problem*

$$\Delta x = f, x_{\partial S} = 0$$

is minimized by the continuous analogue of

$$d(\mathbf{u},\mathbf{v}) := \mathbf{u}^{\mathtt{t}} L\mathbf{v} = \mathbf{f}^{\mathtt{t}} M\mathbf{v} \tag{8}$$

with the mass matrix M providing a scaling of the inner product of f and v. Then M should be (M1) symmetric positive definite, (M2) scale by λ^2, (M3) $M1 = A$. Then $M^{-1}L$ satisfies (L1,L2) and scales like bivariate Laplace-Beltrami.

Finite Elements: For a basis $\{\varphi_m\}$,

$$L_{mn} := \int_S \nabla\varphi_m \nabla\varphi_n dA, \qquad M_{mn} := \int_S \varphi_m\varphi_n dA. \tag{9}$$

Linear basis elements can for example be *hat functions* (vertex-based Lagrange functions) or *Crouzeix-Raviart* functions (mid-edge connected, non-conforming).

Exercise: check that (M3) holds for hat functions, i.e. (crytically, explained in class) $\int [1,0,0][0,1,0]dA = \sum_m M_{mn} = \frac{1}{3}A$.

We check (for the triangle $\triangle(0,\mathbf{m},\mathbf{n})$ with opening angle α and neighbor triangle acros $\mathbf{m},\mathbf{n}$ with angle β) that

$$L_{mn} = -\frac{1}{2}(\cot\alpha + \cot\beta) \tag{10}$$

Abbreviate $\mathtt{k} := \|\mathbf{m} - \frac{\mathbf{mn}}{\mathbf{nn}}\mathbf{n}\|$

$$\begin{pmatrix} 0 \\ \frac{1-0}{\mathtt{k}} \end{pmatrix}^{\mathtt{t}} \begin{pmatrix} \|\mathbf{n}\| \\ \frac{-\frac{\mathbf{mn}}{\mathbf{nn}}}{\mathtt{k}} \end{pmatrix} = ... = \frac{\cos a}{\sin a}. \tag{11}$$