Chapter 2

Basic Finite Element Concepts

This chapter introduces fundamental finite element principles. The basic theory is described in a very general context, which is independent of the specifics of the differential equations, boundary conditions, and approximating subspaces. Despite this conceptual simplicity, the methods apply to a broad range of applications.

In section 2.1 we explain the classical Ritz–Galerkin scheme for Poisson's equation, which serves as a typical model problem. Section 2.2 gives examples of basis functions, defined on triangular or quadrilateral meshes. These standard finite elements are discussed only very briefly, merely to point out essential differences to the weighted spline bases, introduced in chapter 4. After briefly introducing Sobolev spaces in section 2.3, we formulate in section 2.4 the finite element method in an abstract setting. In particular, we define the concept of ellipticity and prove the Lax–Milgram existence theorem for variational problems. Finally, in section 2.5 we derive two basic error estimates, Céa's inequality, and the duality principle of Aubin and Nitsche.

2.1 Model Problem

To explain the basic finite element idea, we consider Poisson's equation with homogeneous boundary conditions,

$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D, \tag{2.1}$$

for a domain $D \subset \mathbb{R}^m$ as a model problem. This boundary value problem describes a number of physical phenomema (cf., e.g., [30]). A simple two-dimensional example is shown in Figure 2.1. An elastic membrane is fixed at its boundary ∂D and subjected to a vertical force with density f. If the resulting displacement $u(x_1, x_2)$ is small, it can be accurately modeled by Poisson's equation.

Multiplying the differential equation $-\Delta u = f$ by a smooth function v, which vanishes on the boundary, and integrating by parts (cf. A.9), it follows that

$$\int_D \operatorname{grad} u \operatorname{grad} v = \int_D f v, \quad v_{|\partial D} = 0. \tag{2.2}$$

This weak form of Poisson's problem suggests a natural discretization. We approximate the solution u by a linear combination

For example,

$$\operatorname{grad} u = (\partial_1 u, \ldots, \partial_m u)$$

is the gradient of an m-variate function u, and $\Delta = \sum_{\nu=1}^{m} \partial_\nu^2$ is the Laplace operator.

The 2-norm for vectors and matrices is denoted by $\|\cdot\|$. Finally, $\|u\|_\ell$ is the norm of a function u in the Sobolev space $H^\ell(D)$, corresponding to the scalar product $\langle \cdot, \cdot \rangle_\ell$ (cf. section 2.3).

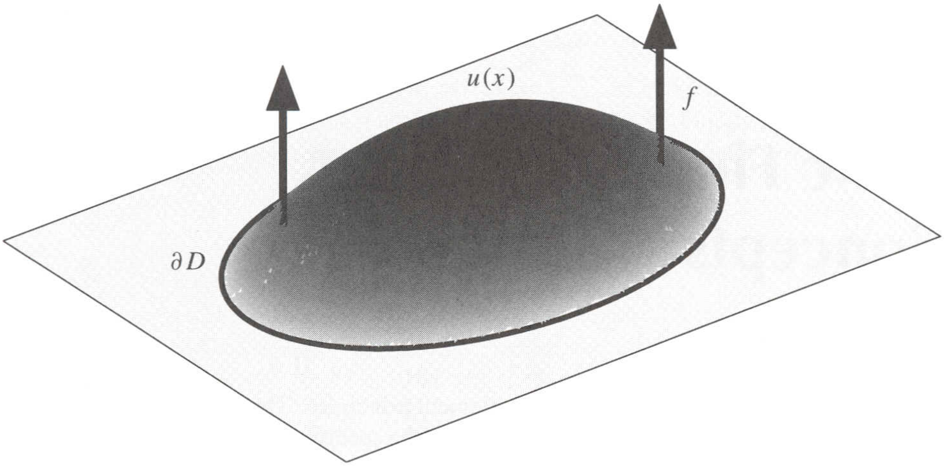

Figure 2.1. *Displacement (magnified) of an elastic membrane.*

$$u_h = \sum_i u_i B_i,$$

of basis functions B_i which satisfy the boundary condition $B_{i|\partial D} = 0$. Usually, the "finite elements" B_i are piecewise polynomials with small support on a mesh of the domain D, and h denotes the maximum diameter of the mesh cells (examples will be given in the next section). Replacing u by u_h in (2.2) and choosing $v = B_k$, we obtain a linear system

$$\int_D \operatorname{grad}\left(\sum_i u_i B_i\right) \operatorname{grad} B_k = \int_D f B_k$$

for the coefficients $U = \{u_i\}$. We summarize this basic finite element scheme, which was first proposed by Ritz [74] and Galerkin [38] at the beginning of the 20th century, as follows.

2.1 Ritz–Galerkin Approximation of Poisson's Problem

The coefficients of a standard finite element approximation

$$u_h = \sum_i u_i B_i, \quad B_{i|\partial D} = 0$$

for the boundary value problem

$$-\Delta u = f, \quad u_{|\partial D} = 0$$

are determined from the linear system $GU = F$ with

$$g_{k,i} = \int_D \operatorname{grad} B_i \operatorname{grad} B_k, \quad f_k = \int_D f B_k.$$

The Ritz–Galerkin method can also be derived via a variational approach. To this end we note that a smooth solution u of Poisson's problem minimizes the energy functional

$$\mathcal{Q}(u) = \frac{1}{2}\int_D \operatorname{grad} u \operatorname{grad} u - \int_D f u \qquad (2.3)$$

over all smooth functions which vanish on ∂D. The characterization of a minimum,

$$\mathcal{Q}(u) \le \mathcal{Q}(u + tv) = \mathcal{Q}(u) + t\left[\int_D \operatorname{grad} u \operatorname{grad} v - f v\right] + \frac{t^2}{2}\int_D \operatorname{grad} v \operatorname{grad} v,$$

where $t \in \mathbb{R}$ is arbitrary, again leads to (2.2). Since the right-hand side is a parabola in t, the expression in square brackets must vanish for all admissible v.

We can define the finite element approximation by minimizing $\mathcal{Q}$ over the linear span

$$\mathbb{B}_h = \operatorname{span}_i B_i$$

of the basis functions B_i. Expanding $\mathcal{Q}(\sum_i u_i B_i)$ yields the quadratic form

$$\mathcal{Q}(u_h) = \frac{1}{2}UGU - FU,$$

which is minimal if $GU = F$.

There is a subtle point hidden in this rather formal argumentation. We have to choose the appropriate class of functions H to ensure that $\inf_{u\in H} \mathcal{Q}(u)$ is attained. Of course, from a numerical point of view, it is legitimate to assume the existence of a smooth solution and to focus entirely on its approximation. However, as we will see in section 2.4, finite element methods and the analysis of variational problems are intimately related. We can establish the existence of weak solutions for very general boundary value problems and, at the same time, prove the solvability of Ritz–Galerkin systems. This will justify the somewhat heuristic arguments for the model problem.

2.2 Mesh-Based Elements

We give in this section a brief overview of some typical classical finite elements. These well-known basis functions will not be used in the remainder of the book, which is devoted to meshless spline approximations. We sketch the standard constructions primarily to compare them with different techniques. They also serve as convenient examples for the abstract variational approach discussed in this chapter.

Most commonly used finite elements are defined on a mesh, i.e., a partition of the domain D into triangles, quadrilaterals, tetrahedra, hexahedra, or other polygonal cells. Triangles and tetrahedra are preferred for most applications since they can be adapted more easily to complicated boundaries. In particular, generating hexahedral meshes in three dimensions is rather difficult. Often one has to resort to mixed partitions in order to overcome the geometric difficulties.

Figure 2.2 shows a triangulation of a two-dimensional domain with the hat-function, the basic piecewise linear finite element. A hat-function B_i equals 1 at an interior vertex x_i and vanishes on all triangles τ not containing x_i. Hence, the graph of B_i is a pyramid