Home Work 1: CAP 6610 Spring ’15
Due Date: Feb 4th 2013
Show all steps. Be as concise as possible.

1. A biased coin lands heads with probability $\frac{1}{5}$ each time it is flipped (i.e. the coin is biased tails). Let $X_1, ..., X_n$ represent $n$ consecutive coin flips with $X_i = 0$ if the coin lands tails and $X_i = 1$ if the coin lands heads. Now, let $\hat{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

(a) Using Markov’s Inequality, give an upper bound on the probability that the coin lands heads at least 60% of the time across $n$ flips. The general form of the Markov’s Inequality is given below.

**Markov’s Inequality**
Suppose that $X$ is a random variable taking only non-negative values. Then, for any $a > 0$ we have

$$P(X \geq a) \leq \frac{E[X]}{a}$$

sol)

$$P(\hat{X}_n \geq 0.6) \leq \frac{E[\hat{X}_n]}{0.6} \leq \frac{0.2}{0.6} \leq \frac{1}{3}$$

(b) Prove following Chebyshev’s Inequality

**Chebyshev’s Inequality**
Let $X$ be a random variable with finite mean $E[X] = \mu$ and $Var[X] = \sigma^2$. Then, for any $k > 0$, we have

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

**Hint**: Use Markov’s Inequality

sol)

By using Markov’s inequality,

$$P(|X - \mu| \geq k) = P(|X - \mu|^2 \geq k^2)$$

$$P(|X - \mu|^2 \geq k^2) \leq \frac{E[(X - \mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$$
(c) Assuming that $X_1, \ldots, X_n$ are iid observations, using Chebyshev’s Inequality, give an upper bound on the probability that the coin lands heads at least 60% of the time across $n$ flips.

$$P(|\hat{X}_n - E[\hat{X}_n]| \geq k) \leq \frac{Var[\hat{X}_n]}{k^2}$$

$$P(|\hat{X}_n - 0.2| \geq 0.4) \leq \frac{Var[\frac{1}{n} \sum_{i=1}^{n} X_i]}{(0.4)^2}$$

$$P(\hat{X}_n \geq 0.6) \leq \frac{\left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} Var[X_i]}{0.16}$$

$$P(\hat{X}_n \geq 0.6) \leq \frac{\left(\frac{1}{n}\right)^2 n 0.16}{0.16}$$

$$P(\hat{X}_n \geq 0.6) \leq \frac{1}{n}$$

2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. We define their

- arithmetic mean: $AM = \frac{x_1 + x_2 + \ldots + x_n}{n}$
- geometric mean: $GM = \sqrt[1][n]{x_1 x_2 \ldots x_n}$
- harmonic mean: $HM = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$

Prove that $AM \geq GM \geq HM$, and the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

(Hint: Use Jensen’s Inequality)

(sol)

By using Jensen’s inequality for the concave function $f(x) = \ln x$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n = \frac{1}{n}$, we can obtain $AM \geq GM$. And by taking $x_k \rightarrow \frac{1}{x_k}$ for $k = 1, 2, \ldots, n$ in $AM \geq GM$ we obtain $GM \geq HM$. 

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